

HOMOGENISATION OF THIN PERIODIC FRAMEWORKS WITH HIGH-CONTRAST INCLUSIONS

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Abstract

We analyse a problem of two-dimensional linearised elasticity for a two-component periodic composite, where one of the components consists of disjoint soft inclusions embedded in a rigid framework. We consider the case when the contrast between the elastic properties of the framework and the inclusions, as well as the ratio between the period of the composite and the framework thickness increase as the period of the composite becomes smaller. We show that in this regime the elastic displacement converges to the solution of a special two-scale homogenised problem, where the microscopic displacement of the framework is coupled both to the slowly-varying “macroscopic” part of the solution and to the displacement of the inclusions. We prove the convergence of the spectra of the corresponding elasticity operators to the spectrum of the homogenised operator with a band-gap structure.

Keywords: Partial differential equations; Periodic homogenisation; Thin structures; Loss of uniform ellipticity; High-contrast; Two-scale convergence; Limit spectrum; Band-gap spectrum.

Introduction

The multi-scale extension of the notion of the weak L^2 -limit was proposed in [11], [2], where a general theorem about two-scale compactness of L^2 -bounded sequences was proved and a corrector-type result for the uniformly elliptic periodic homogenisation problem was established. Multi-scale convergence has proved to be an effective tool in the analysis of the behaviour of periodic composite media under minimal spatial regularity assumptions on the material properties of the composite, *e.g.* measurability and boundedness. Further, in problems where solutions do not converge in the strong L^2 -sense, for example in the presence of degeneracies, see *e.g.* [14], the related techniques have the additional benefit of capturing the multi-scale structure of the limit, by providing a suitable generalised notion of strong convergence. As opposed to the uniformly elliptic case, where the limit function only depends on the macroscopic variable and is a solution to a single boundary-value problem, the multi-scale limit for degenerate homogenisation problems satisfies a coupled system of equations for the macroscopic and microscopic parts of the limit solution. This happens to be the case for periodic “thin structures”, which are the subject of the present work.

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We define a *thin structure* as an arrangement of rods of thickness $a > 0$ joined together at a number of junction points (“nodes”). Fig. 1 shows an example of a thin structure, where the two panels show rods and the “singular” structure obtained by taking the mid-lines of the rods (right). In the literature, equations of elasticity on thin structures are studied by treating it as a parameter $a(\varepsilon)$ linked to the typical rod length ε . In the context of homogenisation, the rods are often assumed to be arranged periodically with period ε , and the asymptotic behaviour of the structure is studied as $\varepsilon \rightarrow 0$. The use of two-scale convergence for the study of periodic singular structures has been proposed in [16, 4], where the two-scale approach of [11, 2] was extended to the setting of general Borel measures, and conditions on the measure sufficient for passing to the two-scale limit were determined.

The use of multi-scale convergence techniques for the analysis of periodic thin structures was initiated in the work [16], which showed that if the thickness of the rods $a = a(\varepsilon)$ is a function of the period ε of the network, such that $\lim_{\varepsilon \rightarrow 0} a(\varepsilon) = 0$, then the overall limit behaviour of the framework depends on the asymptotics of the ratio a/ε^2 as $\varepsilon \rightarrow 0$. In particular, in the case when $\lim_{\varepsilon \rightarrow 0} a/\varepsilon^2 = \theta > 0$, sequences of symmetric gradients of the solutions are, in general, not compact with respect to strong two-scale convergence. As a consequence, the equation describing the limit energy balance is no longer obtained by setting the test function to be the solution of the homogenised equation for the corresponding “singular structure”, obtained by considering the mid-lines of the rods with the measure induced by the thin structure (*cf.* Fig. 1). This problem was addressed by [20], where the correct form of the energy equality was determined and the limit system of equations was derived. This study was followed by the analysis of Sobolev spaces for a variable measure [18, 13], Korn inequalities for periodic frames [19], and gaps in the spectrum of the elasticity operator on a high-contrast periodic structure [21] with non-vanishing volume fraction of the components as $\varepsilon \rightarrow 0$. In the paper [21], which can be viewed as the development of the results of [15] to the high-contrast elasticity context, the band-gap nature of the spectrum of the limit operator is analysed and the convergence of the spectra of the heterogeneous problems to the limit spectrum is proved. Notably, as was first observed by [22], the spectrum of the limit problem for a thin structure in the case of the above “critical” scaling $\lim_{\varepsilon \rightarrow 0} a/\varepsilon^2 = \theta > 0$ shows a remarkable similarity to the limit spectrum for the high-contrast, fixed-volume-fraction case of [21]. Some reasons for this similarity have been found in a recent work [6], which uses operator-theoretic tools to show that the resolvents of both models are operator-norm close to a limit Kronig-Penney model of the so-called “ δ' -type”.

In the present work we consider a two-component periodic composite where the region occupied by the main material (“matrix”) is a framework with $a/\varepsilon^2 \rightarrow \theta > 0$, and the complementary part of the space consisting of disjoint “inclusions” is filled by a less rigid material, so that the ratio between the stiffness of inclusions and the matrix is of the order $O(\varepsilon^2)$. In other words, in addition to the assumption of high contrast, *cf.* [15], we assume that the stiff component is a thin structure so that its volume fraction is of the order $O(\varepsilon)$. While our analysis uses some elements of both multi-scale approaches to thin structures of [16, 20] and high-contrast structures of [15, 21], the proofs of our results, namely homogenisation (Theorem 3.1) and spectral convergence (Theorem 4.1), require new tools that link the behaviour of solutions to the original sequence of problems with rapidly oscillating coefficients on the matrix and on the inclusions. The limit functions for the restrictions of the solutions to each of the two components are coupled together as described in Section 3.1, and lead to a homogenised system of equations of a new kind.

1 Problem formulation and main result

We consider a periodic rod framework (“stiff” component of the composite) filled by a different material (“soft” component). We assume that the rod thickness $a > 0$ is a function of the period $\varepsilon > 0$, and consider the regime when $\lim_{\varepsilon \rightarrow 0} a/\varepsilon^2 = \theta > 0$. The ratio of the elastic moduli of the soft and stiff component is assumed to be of the order $O(\varepsilon^2)$. Denote by F_1^h the domain occupied by the scaled rods of thickness $h := a/\varepsilon$ in the scaled structure of period $Q := [0, 1]^2$ and by F_1 the corresponding singular structure, also of period Q , obtained in the limit $h \rightarrow 0$. The original rod framework is then the

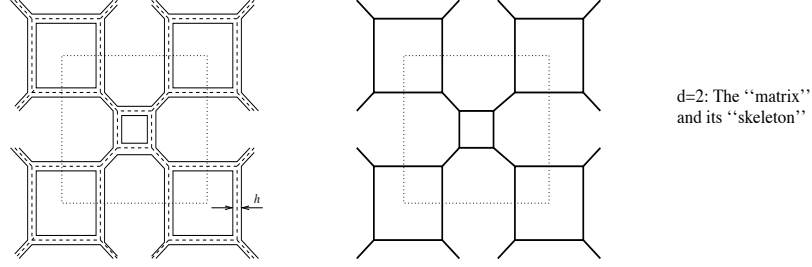


Figure 1: Example of a periodic network and unit cell.

“contraction” $F_1^{h,\varepsilon} := \varepsilon F_1^h$ of the framework F_1^h . The scaled soft component $\mathbb{R}^2 \setminus F_1^h$ and the original soft component $\varepsilon(\mathbb{R}^2 \setminus F_1^h)$ are denoted by F_0^h and $F_0^{h,\varepsilon}$, respectively. We denote by χ_1^h , $\chi_1^{h,\varepsilon}$ and χ_0^h , $\chi_0^{h,\varepsilon}$ the characteristic functions of the respective sets.

In what follows, we consider equations of two-dimensional elasticity in \mathbb{R}^2 . These are obtained from the full system of linearised elasticity in three dimensions when there is a direction, say x_3 , along which material properties are constant, assuming that the displacement does not depend on x_3 . At each point $\mathbf{x} \in \mathbb{R}^2$, the fourth order tensor of the elastic moduli of the medium is set to be given by

$$A^\varepsilon = \varepsilon^2 A_0 \chi_0^h(\cdot/\varepsilon) + A_1 \chi_1^h(\cdot/\varepsilon),$$

where A_0 and A_1 are constant positive definite matrices.¹ For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, we denote by $\Omega_1^{\varepsilon,h} := \Omega \cap F_1^{h,\varepsilon}$ the stiff component and by $\Omega_0^{\varepsilon,h} := \Omega \cap F_0^{h,\varepsilon}$ the stiff component of the composite medium in Ω . Consider the measures λ , λ^h defined on Q by

$$\lambda(B) = \frac{\mathcal{H}^1(F_1 \cap B)}{\mathcal{H}^1(F_1 \cap Q)}, \quad \lambda^h(B) = \frac{\mathcal{H}^2(F_1^h \cap B)}{\mathcal{H}^2(F_1^h \cap Q)} \quad \forall \text{ Borel } B \subset Q,$$

where \mathcal{H}^d , $d = 1, 2$, is the d -dimensional Hausdorff measure (see *e.g.* [9]), and extended to \mathbb{R}^2 by Q -periodicity. Clearly, the weak convergence $\lambda^h \rightharpoonup \lambda$ holds as $h \rightarrow 0$, *i.e.* one has²

$$\lim_{h \rightarrow 0} \int_Q \varphi d\lambda^h = \int_Q \varphi d\lambda \quad \forall \varphi \in [C_{\text{per}}^\infty(Q)]^2.$$

Similarly, for the “composite” measures $\mu := (1/2)d\mathbf{x} + (1/2)\lambda$ and $\mu^h := (1/2)d\mathbf{x} + (1/2)\lambda^h$, where $d\mathbf{x}$ is the plane Lebesgue measure, one has $\mu^h \rightharpoonup \mu$ as $h \rightarrow 0$. Further, we consider the “scaled” measure $\lambda_\varepsilon^h(B) := \varepsilon^2 \lambda^h(\varepsilon^{-1}B)$ for all Borel $B \subset \mathbb{R}^2$, and $\mu_\varepsilon^h := (1/2)d\mathbf{x} + (1/2)\lambda_\varepsilon^h$, so that $\mu_\varepsilon^h \rightharpoonup d\mathbf{x}$ as $\varepsilon \rightarrow 0$.

For $\varepsilon, h > 0$ and $\mathbf{f} \in [L^2(\Omega)]^2$, we look for $\mathbf{u} = \mathbf{u}_\varepsilon^h \in [H_0^1(\Omega)]^2$ such that

$$\begin{aligned} \int_{\Omega_1^{\varepsilon,h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\varphi) d\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\varphi) d\mu_\varepsilon^h \\ + \int_\Omega \mathbf{u}_\varepsilon^h \cdot \varphi d\mu_\varepsilon^h = \int_\Omega \mathbf{f} \cdot \varphi d\mu_\varepsilon^h \quad \forall \varphi \in [H_0^1(\Omega)]^2. \end{aligned} \quad (1.1)$$

¹The scalar product of two symmetric matrices $\boldsymbol{\xi} = \{\xi_{ij}\}_{i,j=1}^2$ and $\boldsymbol{\eta} = \{\eta_{ij}\}_{i,j=1}^2$ is defined by $\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \xi_{ij}\eta_{ij}$. The product of the fourth-order elasticity tensor A with a symmetric matrix $\boldsymbol{\xi}$ is defined as $A\boldsymbol{\xi} = a_{ijkl}\xi_{kl}$ and thus $A\boldsymbol{\xi} \cdot \boldsymbol{\xi} = a_{ijkl}\xi_{ij}\xi_{kl}$.

²We attach the superscript “per” to the notation for a function space when we refer to its subspace of Q -periodic functions.

Define a bilinear form $\mathfrak{B}_\varepsilon^h(\cdot, \cdot)$ and a linear form $\mathfrak{L}_\varepsilon^h(\cdot)$ by

$$\mathfrak{B}_\varepsilon^h(\mathbf{u}, \mathbf{v}) := \int_{\Omega_1^{\varepsilon, h}} A_1 \mathbf{e}(\mathbf{u}) \cdot \mathbf{e}(\mathbf{v}) d\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}) \cdot \mathbf{e}(\mathbf{v}) d\mu_\varepsilon^h + \int_\Omega \mathbf{u} \cdot \mathbf{v} d\mu_\varepsilon^h, \quad \mathfrak{L}_\varepsilon^h(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mu_\varepsilon^h. \quad (1.2)$$

Notice that $\mathfrak{B}_\varepsilon^h$ is coercive and continuous, and $\mathfrak{L}_\varepsilon^h$ is continuous on $[H_0^1(\Omega)]^2$. It is a consequence of the Lax-Milgram lemma (see *e.g.* [8, Chapter 6]) that (1.1) has a unique solution \mathbf{u}_ε^h . In what follows we aim to describe the structure of the limit problem for the weak two-scale limit of the function \mathbf{u}_ε^h as $\varepsilon \rightarrow 0$.

In the theory of homogenisation for periodic rod structures, when A_0 is formally replaced by zero in (1.1), (1.2), the following results hold regardless of the asymptotic behaviour of the ratio a/ε^2 , see [16], [20]:

1. There exists a vector function $\mathbf{u}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega, L_{\text{per}}^2(Q, d\lambda))]^2$ such that:

$$\text{a) } \quad \frac{1}{|\Omega_1^{h, \varepsilon}|} \int_{\Omega_1^{h, \varepsilon}} \mathbf{u}_\varepsilon^h(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}, \mathbf{x}/\varepsilon) d\mathbf{x} \xrightarrow{\varepsilon \rightarrow 0} \int_\Omega \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) d\lambda(\mathbf{y}) d\mathbf{x} \quad \forall \boldsymbol{\varphi} \in [L^2(\Omega, L_{\text{per}}^2(Q, d\mu))]^2; \quad (1.3)$$

$$\text{b) } \quad \frac{1}{|\Omega_1^{h, \varepsilon}|} \int_{\Omega_1^{h, \varepsilon}} |\mathbf{u}_\varepsilon^h(\mathbf{x})|^2 d\mathbf{x} \xrightarrow{\varepsilon \rightarrow 0} \int_\Omega \int_Q |\mathbf{u}(\mathbf{x}, \mathbf{y})|^2 d\lambda(\mathbf{y}) d\mathbf{x}. \quad (1.4)$$

2. The vector $\mathbf{u}(\mathbf{x}, \cdot)$ is a “periodic rigid displacement” (see Definition 2.3). For many frameworks of interest this implies that

$$\mathbf{u}(\mathbf{x}, \mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \boldsymbol{\chi}(\mathbf{x}, \mathbf{y}), \quad \text{a.e. } \mathbf{x} \in \Omega, \quad \lambda\text{-a.e. } \mathbf{y} \in F_1 \cap Q. \quad (1.5)$$

where $\mathbf{u}_0 \in [H_0^1(\Omega)]^2$ and $\boldsymbol{\chi}(\mathbf{x}, \cdot)$ is a “periodic transverse displacement”.

3. The “macroscopic” equation

$$-\text{div}(A_\lambda^{\text{hom}} \mathbf{e}(\mathbf{u}_0)) + \int_Q \mathbf{u}(\cdot, \mathbf{y}) d\lambda(\mathbf{y}) = \mathbf{f} \quad (1.6)$$

holds, where A_λ^{hom} is the “ λ -homogenised tensor” defined by (2.12).

Our main result, Theorem 3.1, states that in the case when $a/\varepsilon^2 \rightarrow \theta > 0$ as $\varepsilon \rightarrow 0$, the solutions \mathbf{u}_ε^h to the problems (1.1), where $h = a/\varepsilon$, converge in an appropriate two-scale sense (see Section 2) to a function $\mathbf{u}(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in \Omega$, $\mathbf{y} \in Q$, whose trace on $F_1 \cap Q$ has the form (1.5) and satisfies an equation involving F_1 -transversal components of the \mathbf{y} -gradient of the function $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{y})$. In addition, the function $\mathbf{u}(\mathbf{x}, \cdot) - \mathbf{u}_0(\mathbf{x}) =: \mathbf{U}(\mathbf{x}, \cdot)$, $\mathbf{x} \in \Omega$, belongs to the space $[H_{\text{per}}^1(Q)]^2$ a.e. $\mathbf{x} \in \Omega$ and satisfies an elliptic equation that couples its values to the solution \mathbf{u}_0 of (1.6), where the average $\int_Q \mathbf{u}(\cdot, \mathbf{y}) d\lambda(\mathbf{y})$ is replaced by $\int_Q \mathbf{u}(\cdot, \mathbf{y}) d\mu(\mathbf{y})$.

More precisely, for each link I of the network F_1 , let $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ be unit tangent and normal vectors that form a positively orientated system. Then all vectors $\mathbf{v} \in \mathbb{R}^2$ are written as $\mathbf{v} = v^{(\tau)} \boldsymbol{\tau} + v^{(\nu)} \boldsymbol{\nu}$, where $v^{(\tau)} = \mathbf{v} \cdot \boldsymbol{\tau}$ and $v^{(\nu)} = \mathbf{v} \cdot \boldsymbol{\nu}$. In Section 3 the vectors \mathbf{U} and $\boldsymbol{\chi}$ are shown to satisfy a system of equations of the form

$$\mathcal{A}_0 \mathbf{U} + \mathbf{u} = \mathbf{f}, \quad \mathcal{L}_\tau \boldsymbol{\chi}^{(\nu)} + \mathcal{T}_\nu U^{(\nu)} + u^{(\nu)} = f^{(\nu)}, \quad (1.7)$$

where \mathcal{A}_0 is a second-order differential operator in Q expressed in terms of the tensor A_0 only, \mathcal{L}_τ is a fourth-order differential operator in the “longitudinal” direction $\boldsymbol{\tau}$, and \mathcal{T}_ν is a first-order differential operator in the “transverse” direction $\boldsymbol{\nu}$ corresponding to each link I .

2 Two-scale structure of solution sequences

In this section we establish the structure of various two-scale limits on the soft and stiff components. This is achieved by taking the limits, as $\varepsilon \rightarrow 0$, of the integrals entering the identity (1.1), with suitably chosen test functions φ .

2.1 Two-scale convergence: definition and properties

We first recall the notion of weak and strong two-scale convergence and their basic properties, see [16]. Within this section $d = 2$ or $d = 3$, and the measure sequence μ_ε^h (“limit” measure μ) can be replaced by the sequence λ_ε^h (“limit” measure λ).

Definition 2.1 (Weak two-scale convergence). Suppose that h is a function of ε and $\{\mathbf{u}_\varepsilon^h\} \subset [L^2(\Omega, d\mu_\varepsilon^h)]^d$ is a bounded sequence:

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} |\mathbf{u}_\varepsilon^h|^2 d\mu_\varepsilon^h < \infty. \quad (2.1)$$

We refer to $\mathbf{u}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^d$ as the *weak two-scale limit* of \mathbf{u}_ε^h , denoted $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$, if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{u}_\varepsilon^h(\mathbf{x}) \cdot \Phi(\mathbf{x}, \mathbf{x}/\varepsilon) d\mu_\varepsilon^h = \int_{\Omega} \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \Phi(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) d\mathbf{x} \quad \forall \Phi \in [L^2(\Omega, C_{\text{per}}(Q))]^d. \quad (2.2)$$

Proposition 2.1 (Two-scale compactness). *If a sequence \mathbf{u}_ε^h is bounded in $[L^2(\Omega, d\mu_\varepsilon^h)]^d$, then it is compact with respect to weak two-scale convergence.*

Proposition 2.2. *If $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$ then $\|\mathbf{u}\|_{[L^2(\Omega \times Q)]^d} \leq \liminf_{\varepsilon \rightarrow 0} \|\mathbf{u}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^d}$.*

Definition 2.2. Let \mathbf{u}_ε^h be a bounded sequence in $[L^2(\Omega, d\mu_\varepsilon^h)]^d$. We say that a function $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^d$ is the *strong two-scale limit* of \mathbf{u}_ε^h , denoted $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$, if for any weakly two-scale convergent sequence $\mathbf{v}_\varepsilon^h \xrightarrow{2} \mathbf{v}$ one has

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{u}_\varepsilon^h \cdot \mathbf{v}_\varepsilon^h d\mu_\varepsilon^h = \int_{\Omega} \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v}(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) d\mathbf{x}. \quad (2.3)$$

Note that by setting $\mathbf{v}_\varepsilon^h = \mathbf{u}_\varepsilon^h$ one has

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\mathbf{u}_\varepsilon^h|^2 d\mu_\varepsilon^h = \int_{\Omega} \int_Q |\mathbf{u}|^2 d\mu d\mathbf{x}. \quad (2.4)$$

The next proposition shows that the converse also holds.

Proposition 2.3. *If $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$ and the convergence (2.4) holds, then $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$.*

Proposition 2.4. *For any arbitrary $a \in L^\infty(Q)$, the weak (resp. strong) two-scale convergence of \mathbf{u}_ε^h to $\mathbf{u}(\mathbf{x}, \mathbf{y})$ implies the weak (resp. strong) two-scale convergence of $a(\cdot/\varepsilon)\mathbf{u}_\varepsilon^h$ to $a(\mathbf{y})\mathbf{u}(\mathbf{x}, \mathbf{y})$.*

2.2 Two-scale compactness of solutions to (1.1)

Consider the equation (1.1) with $\varphi = \mathbf{u}_\varepsilon^h$:

$$\int_{\Omega_1^{\varepsilon, h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) d\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) d\mu_\varepsilon^h + \int_{\Omega} |\mathbf{u}_\varepsilon^h|^2 d\mu_\varepsilon^h = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_\varepsilon^h d\mu_\varepsilon^h. \quad (2.5)$$

Using ellipticity estimates on the left-hand side and the inequality $2ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$, on the right-hand side yields

$$c_0 \varepsilon^2 \int_{\Omega_0^{\varepsilon, h}} |\mathbf{e}(\mathbf{u}_\varepsilon^h)|^2 d\mu_\varepsilon^h + c_1 \int_{\Omega_1^{\varepsilon, h}} |\mathbf{e}(\mathbf{u}_\varepsilon^h)|^2 d\mu_\varepsilon^h + \frac{1}{2} \int_{\Omega} |\mathbf{u}_\varepsilon^h|^2 d\mu_\varepsilon^h \leq \frac{1}{2} \int_{\Omega} |\mathbf{f}|^2 d\mu_\varepsilon^h,$$

where c_0, c_1 are the ellipticity constants of A_0, A_1 . Hence, the following *a priori* bounds hold.

Proposition 2.5. *Let \mathbf{u}_ε^h be a sequence in $[L^2(\Omega, d\mu_\varepsilon^h)]^2$ of solutions to (1.1). Then there exists $C > 0$ such that*

$$\|\mathbf{u}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} \leq C, \quad \|\mathbf{e}(\mathbf{u}_\varepsilon^h)\|_{[L^2(\Omega_1^{\varepsilon, h}, d\mu_\varepsilon^h)]^3} \leq C, \quad \varepsilon \|\mathbf{e}(\mathbf{u}_\varepsilon^h)\|_{[L^2(\Omega_0^{\varepsilon, h}, d\mu_\varepsilon^h)]^3} \leq C.$$

Using two-scale compactness of L^2 -bounded sets (see Proposition 2.1), we assume that the sequences

$$\mathbf{u}_\varepsilon^h, \quad \chi_1^{h, \varepsilon} \mathbf{u}_\varepsilon^h \quad (\text{displacements}), \quad \text{and} \quad \chi_1^{h, \varepsilon} \mathbf{e}(\mathbf{u}_\varepsilon^h), \quad \varepsilon \chi_0^{h, \varepsilon} \mathbf{e}(\mathbf{u}_\varepsilon^h) \quad (\text{strains})$$

weakly two-scale converge to functions $\mathbf{u}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2$, $\hat{\mathbf{u}}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega \times Q, d\mathbf{x} \times d\lambda)]^2$ (displacements), and $\mathbf{p}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega \times Q, d\mathbf{x} \times d\lambda)]^3$, $\hat{\mathbf{p}}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega \times Q, d\mathbf{x} \times d\mathbf{y})]^3$ (strains), respectively. Here, each of the spaces $L^2(\Omega \times Q, d\mathbf{x} \times d\lambda)$ and $L^2(\Omega \times Q, d\mathbf{x} \times d\mathbf{y})$ is treated as a subspace of $L^2(\Omega \times Q, d\mu)$.

2.3 Rigid displacements, potential and solenoidal matrices

Definition 2.3. A vector function $\mathbf{u} \in [L^2_{\text{per}}(Q, d\lambda)]^2$ is said to be a *periodic rigid displacement* (with respect the measure λ) if there exists a sequence $\{\mathbf{u}_n\} \subset [C^\infty_{\text{per}}(Q)]^2$ such that $(\mathbf{u}_n, \mathbf{e}(\mathbf{u}_n)) \rightarrow (\mathbf{u}, 0)$ in $[L^2_{\text{per}}(Q, d\lambda)]^5$. We denote the set of periodic rigid displacements by \mathcal{R} , omitting the reference to the measure λ .

We assume (see *e.g.* [16] for relevant examples of periodic frameworks) that any $\mathbf{u} \in \mathcal{R}$ has a unique representation

$$\mathbf{u}(\mathbf{y}) = \mathbf{c} + \boldsymbol{\chi}(\mathbf{y}), \quad \mathbf{y} \in Q, \quad (2.6)$$

where $\mathbf{c} \in \mathbb{R}^2$ and $\boldsymbol{\chi}$ is a periodic transverse displacement, *i.e.* on each link of the singular network F_1 it is orthogonal to the link. Thus \mathcal{R} is the direct sum of \mathbb{R}^2 and the set of transverse displacements, which we denote by $\hat{\mathcal{R}}$. The next definition characterises transverse displacements that occur in the study of rod networks with $a/\varepsilon^2 \rightarrow \theta > 0$ as $\varepsilon \rightarrow 0$.

Definition 2.4. Denote by I_1, \dots, I_n the links of the network F_1 sharing an arbitrary node \mathcal{O} , and denote by $(\boldsymbol{\chi} \cdot \boldsymbol{\nu})'$ the derivative in the tangential direction: $(\boldsymbol{\chi} \cdot \boldsymbol{\nu})' := (\boldsymbol{\tau} \cdot \nabla)(\boldsymbol{\chi} \cdot \boldsymbol{\nu})$. The set $\hat{\mathcal{R}}^0 \subset \hat{\mathcal{R}}$ is defined to consist of periodic transverse displacements $\boldsymbol{\chi}$ satisfying the following conditions:

(C1) The function $\boldsymbol{\chi} \cdot \boldsymbol{\nu}_j|_{I_j}$, $j = 1, 2, \dots, n$, has square integrable second derivatives on I_j , *i.e.* one has $\boldsymbol{\chi} \cdot \boldsymbol{\nu} \in H^2(I_j)$.

(C2) The first derivative along the link is continuous across each node: $(\boldsymbol{\chi} \cdot \boldsymbol{\nu}_1)'|_{\mathcal{O}} = (\boldsymbol{\chi} \cdot \boldsymbol{\nu}_2)'|_{\mathcal{O}} = \dots = (\boldsymbol{\chi} \cdot \boldsymbol{\nu}_n)'|_{\mathcal{O}}$.

(C3) Each node is fastened: $\boldsymbol{\chi}|_{\mathcal{O}} = \mathbf{0}$.

The norm in $\hat{\mathcal{R}}^0$ is defined to be the sum of the H^2 -norms of $\boldsymbol{\chi} \cdot \boldsymbol{\nu}$ over all the links.

Definition 2.5. For a given Borel measure \varkappa on Q , we define the space V_{pot}^κ of \varkappa -potential matrices as the closure of the set $\{\mathbf{e}(\mathbf{u}) | \mathbf{u} \in [C^\infty_{\text{per}}(Q)]^2\}$ in the space $[L^2_{\text{per}}(Q, d\varkappa)]^3$. A symmetric matrix $\mathbf{v} \in [L^2_{\text{per}}(Q, d\varkappa)]^3$ is said to be \varkappa -solenoidal if

$$\int_Q \mathbf{v} \cdot \mathbf{e}(\mathbf{u}) d\varkappa = 0 \quad \forall \mathbf{u} \in [C^\infty_{\text{per}}(Q)]^2.$$

Denoting by V_{sol}^κ the set of κ -solenoidal matrices, we can write (see *e.g.* [16]) $[L_{\text{per}}^2(Q, d\kappa)]^3 = V_{\text{pot}}^\kappa \oplus V_{\text{sol}}^\kappa$. It follows that the orthogonal decomposition $[L^2(\Omega \times Q, d\mathbf{x} \times d\kappa)]^3 = L^2(\Omega, V_{\text{pot}}^\kappa) \oplus L^2(\Omega, V_{\text{sol}}^\kappa)$ holds, where the *two-scale* L^2 -spaces of κ -potential and κ -solenoidal vector fields are the closures of the linear spans of matrices $w\mathbf{e}(\mathbf{u})$, $w \in C_0^\infty(\Omega)$, $\mathbf{u} \in [C_{\text{per}}^\infty(Q)]^2$ and $w\mathbf{v}$, $w \in C_0^\infty(\Omega)$, $\mathbf{v} \in V_{\text{sol}}^\kappa$, with respect to the norm of $[L^2(\Omega \times Q, d\mathbf{x} \times d\kappa)]^3$. When κ is the Lebesgue measure on Q , we simply write V_{pot} , V_{sol} , $[L^2(\Omega \times Q)]^3$.

2.4 Convergence on the stiff component

We first study the relationship between the limit functions $\mathbf{u}(\mathbf{x}, \mathbf{y})$ and $\hat{\mathbf{u}}(\mathbf{x}, \mathbf{y})$, see Section 2.2.

Definition 2.6. Denote $\psi_\varepsilon^h := \psi^h(\cdot/\varepsilon)$, where $\psi^h \in [L_{\text{per}}^2(Q, d\mu^h)]^2$ extended to \mathbb{R}^2 by Q -periodicity.

1. We say that the sequence ψ_ε^h weakly converges to $\psi \in [L_{\text{per}}^2(Q, d\mu)]^2$, and write $\psi_\varepsilon^h \xrightarrow[\mu_\varepsilon^h]{\mu_\varepsilon^h} \psi$, if

$$\int_Q \psi_\varepsilon^h \cdot \boldsymbol{\xi}(\cdot/\varepsilon) d\mu_\varepsilon^h \longrightarrow \int_Q \psi \cdot \boldsymbol{\xi} d\mu \quad \forall \boldsymbol{\xi} \in [C_{\text{per}}^\infty(Q)]^2,$$

where the test function $\boldsymbol{\xi}$ is extended to \mathbb{R}^2 by Q -periodicity.

2. We say that ψ_ε^h strongly converge to a function $\psi \in [L_{\text{per}}^2(Q, d\mu)]^2$, and write $\psi_\varepsilon^h \xrightarrow[\mu_\varepsilon^h]{\mu_\varepsilon^h} \psi$, if

$$\int_Q \psi_\varepsilon^h \cdot \boldsymbol{\xi}^h(\cdot/\varepsilon) d\mu_\varepsilon^h \longrightarrow \int_Q \psi \cdot \boldsymbol{\xi} d\mu \quad \text{if and only if} \quad \boldsymbol{\xi}_\varepsilon^h \xrightarrow[\mu_\varepsilon^h]{\mu_\varepsilon^h} \boldsymbol{\xi}.$$

Proposition 2.6. If $\mathbf{u}_\varepsilon^h(\mathbf{x}) \xrightarrow[\mu_\varepsilon^h]{\mu_\varepsilon^h} \mathbf{u}(\mathbf{x}, \mathbf{y})$ and $\psi_\varepsilon^h \xrightarrow[\mu_\varepsilon^h]{\mu_\varepsilon^h} \psi$, then

$$\int_\Omega \mathbf{u}_\varepsilon^h \cdot \psi_\varepsilon^h \varphi d\mu_\varepsilon^h \longrightarrow \int_\Omega \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \psi(\mathbf{y}) \varphi(\mathbf{x}) d\mu(\mathbf{y}) d\mathbf{x} \quad \forall \varphi \in C_0^\infty(\Omega).$$

Proof. Since $\psi_\varepsilon^h \xrightarrow[\mu_\varepsilon^h]{\mu_\varepsilon^h} \psi$, it follows that for all $\boldsymbol{\zeta} \in [C_{\text{per}}(Q)]^2$ the relation

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega |\psi_\varepsilon^h - \boldsymbol{\zeta}(\cdot/\varepsilon)|^2 d\mu_\varepsilon^h = |\Omega| \int_Q |\psi - \boldsymbol{\zeta}|^2 d\mu \quad (2.7)$$

holds. Notice further that, by the Hölder inequality, one has

$$\left| \int_\Omega \mathbf{u}_\varepsilon^h \cdot (\psi_\varepsilon^h - \boldsymbol{\zeta}(\cdot/\varepsilon)) \varphi d\mu_\varepsilon^h \right| \leq \max_\Omega |\varphi| \|\mathbf{u}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} \left(\int_\Omega |\psi^h - \boldsymbol{\zeta}(\cdot/\varepsilon)|^2 d\mu_\varepsilon^h \right)^{1/2}.$$

The weak two-scale convergence of \mathbf{u}_ε^h and the relation (2.7) imply that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| \int_\Omega \mathbf{u}_\varepsilon^h \cdot \psi_\varepsilon^h \varphi d\mu_\varepsilon^h - \int_\Omega \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\zeta}(\mathbf{y}) \varphi(\mathbf{x}) d\mu(\mathbf{y}) d\mathbf{x} \right| \\ &= \limsup_{\varepsilon \rightarrow 0} \left| \int_\Omega \mathbf{u}_\varepsilon^h \cdot \psi_\varepsilon^h \varphi d\mu_\varepsilon^h - \int_\Omega \mathbf{u}_\varepsilon^h \cdot \boldsymbol{\zeta}(\cdot/\varepsilon) \varphi d\mu_\varepsilon^h \right| \leq C \left(\int_Q |\psi - \boldsymbol{\zeta}|^2 d\mu \right)^{1/2} \quad \forall \boldsymbol{\zeta} \in [C_{\text{per}}(Q)]^2. \end{aligned}$$

The claim now follows by choosing an approximation sequence $\boldsymbol{\zeta} = \boldsymbol{\zeta}_k$ such that $\boldsymbol{\zeta}_k \rightarrow \psi$ in $[L_{\text{per}}^2(Q, d\mu)]^2$. \square

Theorem 2.1. The function $\hat{\mathbf{u}}$ is the trace of \mathbf{u} on F_1 , in the sense that $\mathbf{u}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{u}}(\mathbf{x}, \mathbf{y})$ a.e. $\mathbf{x} \in \Omega$, λ -a.e. $\mathbf{y} \in F_1$.

Proof. For all functions $\widehat{\psi} \in [L^2_{\text{per}}(Q, d\lambda)]^2$ and $h > 0$, we define

$$\psi(\mathbf{y}) := \begin{cases} \widehat{\psi}(\mathbf{y}), & \mathbf{y} \in F_1 \cap Q, \\ 0, & \mathbf{y} \in Q \setminus F_1, \end{cases} \quad [\psi]^h(\mathbf{y}) := \begin{cases} \sum_{I_j} \widehat{\psi}_j^h(\mathbf{y}), & \mathbf{y} \in F_1^h \cap Q, \\ 0, & \mathbf{y} \in Q \setminus F_1^h, \end{cases} \quad (2.8)$$

where the summation is carried out over all links I_j of $F_1 \cap Q$, and for each link I_j we set $\widehat{\psi}_j^h(\mathbf{y}) = \widehat{\psi}(\mathbf{y}^*)$ whenever \mathbf{y} is in the h -neighbourhood of I_j and $|\mathbf{y} - \mathbf{y}^*| = \text{dist}(\mathbf{y}, I_j)$, $\mathbf{y}^* \in I_j$, and $\widehat{\psi}_j^h(\mathbf{y}) = 0$ otherwise. Notice that for all $\varphi \in C_0^\infty(\Omega)$ one has

$$\int_{\Omega} \mathbf{u}_\varepsilon^h \cdot [\psi]_\varepsilon^h \varphi d\mu_\varepsilon^h = \int_{\Omega} \mathbf{u}_\varepsilon^h \chi_1^h(\cdot/\varepsilon) \cdot [\psi]_\varepsilon^h \varphi d\mu_\varepsilon^h \quad (2.9)$$

Due to the fact that $[\psi]_\varepsilon^h \xrightarrow{\mu_\varepsilon^h} \psi$, the following convergence holds:

$$\int_{\Omega} \mathbf{u}_\varepsilon^h \cdot [\psi]_\varepsilon^h \varphi d\mu_\varepsilon^h \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \psi(\mathbf{y}) \varphi(\mathbf{x}) d\mu(\mathbf{y}) d\mathbf{x} = \frac{1}{2} \int_{\Omega} \int_Q \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \widehat{\psi}(\mathbf{y}) \varphi(\mathbf{x}) d\lambda(\mathbf{y}) d\mathbf{x}. \quad (2.10)$$

Similarly, for the first integral on the right-hand side of (2.9), we obtain

$$\int_{\Omega} \mathbf{u}_\varepsilon^h \chi_1^h(\cdot/\varepsilon) \cdot [\psi]_\varepsilon^h \varphi d\mu_\varepsilon^h \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\Omega} \int_Q \widehat{\mathbf{u}}(\mathbf{x}, \mathbf{y}) \cdot \widehat{\psi}(\mathbf{y}) \varphi(\mathbf{x}) d\lambda(\mathbf{y}) d\mathbf{x}. \quad (2.11)$$

It follows that the limits in (2.10) and (2.11) coincide, as required. \square

The next theorem, proved in [16], describes the structure of the two-scale limit $\widehat{\mathbf{u}}$. Recall that on the stiff component $F_1^{h,\varepsilon}$ the symmetric gradient is bounded and hence $\varepsilon \chi_1^{\varepsilon,h} \mathbf{e}(\mathbf{u}_\varepsilon^h) \rightarrow 0$ in $[L^2(\Omega_1^{\varepsilon,h}, d\lambda_\varepsilon^h)]^3$.

Theorem 2.2. [Theorems 12.2, 12.3 and Lemma 9.6 in [16]]

1. It follows from $\chi_1^{\varepsilon,h} \mathbf{u}_\varepsilon^h \xrightarrow{2} \widehat{\mathbf{u}}(\mathbf{x}, \mathbf{y})$ in $[L^2(\Omega_1^{\varepsilon,h}, d\mu_\varepsilon^h)]^2$ and $\varepsilon \chi_1^{\varepsilon,h} \mathbf{e}(\mathbf{u}_\varepsilon^h) \rightarrow 0$ in $[L^2(\Omega_1^{\varepsilon,h}, d\lambda_\varepsilon^h)]^3$, that $\forall \mathbf{x} \in \Omega$, λ -a.e. $\mathbf{y} \in F_1$ one has $\widehat{\mathbf{u}}(\mathbf{x}, \mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \chi(\mathbf{x}, \mathbf{y})$ where $\mathbf{u}_0 \in [H_0^1(\Omega)]^2$ and $\chi \in L^2(\Omega, \widehat{\mathcal{R}})$.
2. If the convergence $\chi_1^{\varepsilon,h} \mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}_0(\mathbf{x}) + \chi(\mathbf{x}, \mathbf{y})$ holds in $[L^2(\Omega_1^{\varepsilon,h}, d\lambda_\varepsilon^h)]^2$ and the sequence $\{\chi_1^{h,\varepsilon} \mathbf{e}(\mathbf{u}_\varepsilon^h)\}$ is bounded in $[L^2(\Omega_1^{\varepsilon,h}, d\lambda_\varepsilon^h)]^3$, then, up passing to a subsequence, one has $\mathbf{e}(\mathbf{u}_\varepsilon^h) \xrightarrow{2} \mathbf{e}(\mathbf{u}_0(\mathbf{x})) + \mathbf{v}(\mathbf{x}, \mathbf{y})$ in $[L^2(\Omega_1^{\varepsilon,h}, d\lambda_\varepsilon^h)]^3$, where $\mathbf{v}(\mathbf{x}, \mathbf{y}) \in L^2(\Omega, V_{\text{pot}}^\lambda)$.

Under the additional assumption that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_\mathbf{y}(\varphi)(\cdot/\varepsilon) w d\lambda_\varepsilon^h = 0 \quad \forall \varphi \in [C_{\text{per}}^\infty(Q)]^2, \quad w \in C_0^\infty(\Omega),$$

the two-scale convergence $\chi_1^{h,\varepsilon} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \xrightarrow{2} A_1 \{\mathbf{e}(\mathbf{u}_0(\mathbf{x})) + \mathbf{v}(\mathbf{x}, \mathbf{y})\}$ holds in $[L^2(\Omega_1^{\varepsilon,h}, d\lambda_\varepsilon^h)]^3$, where the limit function is an element of $L^2(\Omega, V_{\text{sol}}^\lambda)$.

Remark 1. Define the “ λ -homogenised” tensor A_λ^{hom} by the minimisation problem

$$A_\lambda^{\text{hom}} \boldsymbol{\xi} \cdot \boldsymbol{\xi} = \min_{\mathbf{v} \in V_{\text{pot}}^\lambda} \int_Q A_1(\boldsymbol{\xi} + \mathbf{v}) \cdot (\boldsymbol{\xi} + \mathbf{v}) d\lambda \quad \forall \boldsymbol{\xi} \in \text{Sym}_2, \quad (2.12)$$

where Sym_2 is the space of symmetric (2×2) -matrices. Theorem 2.2 implies that $\chi_1^{h,\varepsilon} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \rightharpoonup A_\lambda^{\text{hom}} \mathbf{e}(\mathbf{u}_0)$ in the sense of the usual weak convergence in $[L^2(\Omega)]^3$.

The description of the structure of the two-scale limit of $\chi_1^{\varepsilon,h} \mathbf{u}_\varepsilon^h$ is a consequence of several statements proved in [20]. Combining this with Theorem 2.1, we obtain the following result (cf. [20, Theorem 3.1]).

Theorem 2.3. In the formula $\widehat{\mathbf{u}}(\mathbf{x}, \mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \chi(\mathbf{x}, \mathbf{y})$, the transverse displacement χ is an element of the space $L^2(\Omega, \widehat{\mathcal{R}}^0)$.

2.5 Convergence on the soft component

Theorem 2.4. *For all sequences $\{\mathbf{u}_\varepsilon^h\} \subset [H^1(\Omega)]^2$ such that $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}(\mathbf{x}, \mathbf{y})$ in $[L^2(\Omega_0^{\varepsilon, h}, d\mu_\varepsilon^h)]^2$ and $\varepsilon \chi_0^{\varepsilon, h} \mathbf{e}(\mathbf{u}_\varepsilon^h) \xrightarrow{2} \tilde{\mathbf{p}}(\mathbf{x}, \mathbf{y})$ in $[L^2(\Omega_0^{\varepsilon, h}, d\mu_\varepsilon^h)]^3$, one has $\mathbf{u} \in [L^2(\Omega, H^1(Q))]^2$ and $\tilde{\mathbf{p}}(\mathbf{x}, \mathbf{y}) = \mathbf{e}_y(\mathbf{u}(\mathbf{x}, \mathbf{y}))$ a.e. $\mathbf{x} \in \Omega, \mathbf{y} \in Q$.*

Proof. For each $\delta > 0$, consider a C^∞ -domain Q_δ such that $F_0^{2\delta} \cap Q \subset Q_\delta \subset F_0^\delta \cap Q$ and the set

$$\mathcal{X}_\delta := \{\mathbf{b} \in [C^\infty(Q_\delta)]^3 : \mathbf{b} \mathbf{n}|_{\partial Q_\delta} = 0\},$$

where \mathbf{n} is the unit normal to ∂Q_δ . For all $\mathbf{b} \in \mathcal{X}_\delta$, $\mathbf{a} = \operatorname{div} \mathbf{b}$ in Q_δ , consider the functions

$$\tilde{\mathbf{a}}(\mathbf{y}) := \begin{cases} \mathbf{a}(\mathbf{y}), & \mathbf{y} \in Q_\delta, \\ 0, & \mathbf{y} \in Q \setminus Q_\delta, \end{cases} \quad \tilde{\mathbf{b}}(\mathbf{y}) := \begin{cases} \mathbf{b}(\mathbf{y}), & \mathbf{y} \in Q_\delta, \\ 0, & \mathbf{y} \in Q \setminus Q_\delta, \end{cases} \quad (2.13)$$

extended to \mathbb{R}^2 by Q -periodicity. Then for sufficiently small $\varepsilon > 0$ (recall that $h \rightarrow 0$ as $\varepsilon \rightarrow 0$) the following identity holds:

$$\varepsilon \int_{\Omega_0^{\varepsilon, h}} \tilde{\mathbf{b}}(\cdot/\varepsilon) \cdot \mathbf{e}(\boldsymbol{\psi}) d\mu_\varepsilon^h = - \int_{\Omega_0^{\varepsilon, h}} \tilde{\mathbf{a}}(\cdot/\varepsilon) \cdot \boldsymbol{\psi} d\mu_\varepsilon^h \quad \forall \boldsymbol{\psi} \in [H_0^1(\Omega)]^2. \quad (2.14)$$

Setting $\boldsymbol{\psi} = \varphi \mathbf{u}_\varepsilon^h$, $\varphi \in C_0^\infty(\Omega)$, in (2.14) yields

$$\varepsilon \int_{\Omega_0^{\varepsilon, h}} \tilde{\mathbf{b}}(\cdot/\varepsilon) \varphi \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) d\mu_\varepsilon^h + \varepsilon \int_{\Omega_0^{\varepsilon, h}} \tilde{\mathbf{b}}(\cdot/\varepsilon) \cdot \frac{1}{2} (\mathbf{u}_\varepsilon^h \otimes \nabla \varphi + \nabla \varphi \otimes \mathbf{u}_\varepsilon^h) d\mu_\varepsilon^h = - \int_{\Omega_0^{\varepsilon, h}} \tilde{\mathbf{a}}(\cdot/\varepsilon) \cdot \varphi \mathbf{u}_\varepsilon^h d\mu_\varepsilon^h.$$

Passing to the limit in the last identity as $\varepsilon \rightarrow 0$ and using the fact that $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$ vanish in $Q \setminus Q_\delta$, we obtain

$$\int_\Omega \int_{Q_\delta} \tilde{\mathbf{p}}(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{x}) \cdot \mathbf{b}(\mathbf{y}) d\mathbf{y} d\mathbf{x} = - \int_\Omega \int_{Q_\delta} \mathbf{u}(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{x}) \cdot \mathbf{a}(\mathbf{y}) d\mathbf{y} d\mathbf{x}.$$

As $\varphi \in C_0^\infty(\Omega)$ is arbitrary, it follows that

$$\int_{Q_\delta} \tilde{\mathbf{p}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{b}(\mathbf{y}) d\mathbf{y} = - \int_{Q_\delta} \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{a}(\mathbf{y}) d\mathbf{y} \quad \text{a.e. } \mathbf{x} \in \Omega. \quad (2.15)$$

Taking divergence-free fields $\mathbf{b} \in \mathcal{X}_\delta$ in (2.15) we infer (see *e.g.* [7]) the existence of $\mathbf{v} \in [L^2(\Omega, H^1(Q_\delta))]^2$ such that $\tilde{\mathbf{p}}(\mathbf{x}, \mathbf{y}) = \mathbf{e}_y(\mathbf{v}(\mathbf{x}, \mathbf{y}))$, $\mathbf{y} \in Q_\delta$, which implies

$$\int_{Q_\delta} \mathbf{v}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{a}(\mathbf{y}) d\mathbf{y} = \int_{Q_\delta} \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{a}(\mathbf{y}) d\mathbf{y} \quad \text{a.e. } \mathbf{x} \in \Omega,$$

$$\forall \mathbf{a} \in \{\operatorname{div} \mathbf{b} \mid \mathbf{b} \in \mathcal{X}_\delta\} = \left\{ \mathbf{a} \in [C^\infty(Q_\delta)]^2 : \int_{Q_\delta} \mathbf{a} = 0 \right\}.$$

Using the density in $[L^2(Q_\delta)]^2$ of vector functions \mathbf{a} having the above representation implies that $\mathbf{v}(\mathbf{x}, \mathbf{y})$ and $\mathbf{u}(\mathbf{x}, \mathbf{y})$ differ by a constant for $\mathbf{y} \in Q_\delta$, hence $\tilde{\mathbf{p}} = \mathbf{e}_y(\mathbf{v}) = \mathbf{e}_y(\mathbf{u})$, a.e. $\mathbf{y} \in Q_\delta$. By virtue of the arbitrary choice of the parameter δ , we conclude that $\tilde{\mathbf{p}} = \mathbf{e}_y(\mathbf{u})$ for a.e. $\mathbf{y} \in Q$. \square

3 Homogenisation theorem

In what follows, we consider the case of the framework F_1 shown in Fig. 2 (“model framework”). However, the analysis presented is readily extended to any framework such that the representation (2.6) for periodic rigid displacements, with obvious modifications in the statements.

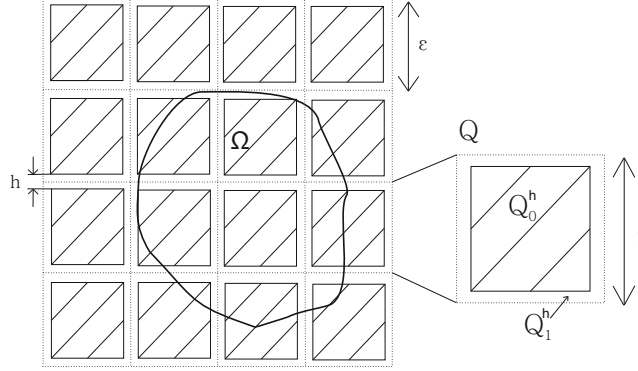


Figure 2: Periodic network with high contrast, where $Q_j^h := F_j^h \cap Q$, $j = 0, 1$.

3.1 Homogenised system of equations

Definition 3.1. We denote by V the *energy space* consisting of vectors

$$\begin{aligned} \mathbf{u}(\mathbf{x}, \mathbf{y}) &= \mathbf{u}_0(\mathbf{x}) + \mathbf{U}(\mathbf{x}, \mathbf{y}), \quad \mathbf{u}_0 \in [H_0^1(\Omega)]^2, \quad \mathbf{U} \in [L^2(\Omega, H_{\text{per}}^1(Q))]^2, \\ \mathbf{U}(\mathbf{x}, \mathbf{y}) &= \chi(\mathbf{x}, \mathbf{y}), \quad \text{a.e. } \mathbf{x} \in \Omega, \quad \lambda\text{-a.e. } \mathbf{y} \in F_1 \cap Q, \quad \chi \in L^2(\Omega, \hat{\mathcal{R}}^0). \end{aligned}$$

We also denote by \mathfrak{H} the closure in of V in $[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2$.

For a given $\mathbf{f} \in [L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2$, we refer to $\mathbf{u} \in V$ as the *solution of the homogenised problem* if

$$\begin{aligned} \int_{\Omega} A_{\lambda}^{\text{hom}} \mathbf{e}(\mathbf{u}_0) \cdot \mathbf{e}(\varphi_0) d\mathbf{x} + \frac{\theta^2}{6} \int_{\Omega} \int_Q K_1 \chi'' \cdot \Phi'' d\lambda d\mathbf{x} + \frac{1}{2} \int_{\Omega} \int_Q A_0 \mathbf{e}_{\mathbf{y}}(\mathbf{U}) \cdot \mathbf{e}_{\mathbf{y}}(\Phi) d\mathbf{y} d\mathbf{x} \\ + \int_{\Omega} \int_Q (\mathbf{u}_0 + \mathbf{U}) \cdot \varphi d\mu d\mathbf{x} = \int_{\Omega} \int_Q \mathbf{f} \cdot \varphi d\mu d\mathbf{x} \quad \forall \varphi(\mathbf{x}, \mathbf{y}) = \varphi_0(\mathbf{x}) + \Phi(\mathbf{x}, \mathbf{y}) \in V, \end{aligned} \quad (3.1)$$

where A_{λ}^{hom} is given by (2.12), and K_1 is a function on the network $F_1 \cap Q$, defined on each link of the network by

$$K_1 := (A_1^{-1} \boldsymbol{\eta} \cdot \boldsymbol{\eta})^{-1}, \quad \boldsymbol{\eta} := \boldsymbol{\tau} \otimes \boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is the tangent to the current link, and the prime denotes the tangential derivative, as in Definition 2.4, *e.g.* $\chi' := (\boldsymbol{\tau} \cdot \nabla \chi) \boldsymbol{\nu}$.

The identity (3.1) is equivalent to a system of partial differential equations, which is obtained by considering various classes of test functions in (3.1). First, taking functions of the form $\varphi(\mathbf{x}, \mathbf{y}) = \varphi_0(\mathbf{x})$ yields (*cf.* (1.6), where $\mathbf{f} \in [L^2(\Omega)]^2$):

$$-\text{div}(A_{\lambda}^{\text{hom}} \mathbf{e}(\mathbf{u}_0)) + \mathbf{u}_0 + \langle \mathbf{U} \rangle = \langle \mathbf{f} \rangle \quad (3.2)$$

where the angle brackets denote macroscopic averaging, *e.g.* $\langle \mathbf{U} \rangle := \int_Q \mathbf{U}(\cdot, \mathbf{y}) d\mu(\mathbf{y})$.

Further, we set $\varphi(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}) \Psi(\mathbf{y})$, with $\varphi \in C_0^{\infty}(\Omega)$, $\Psi \in \tilde{V}$, where the space \tilde{V} consists of functions in $[H_{\text{per}}^1(Q)]^2$ whose trace on $F_1 \cap Q$ coincides with a rigid-body motion λ -a.e. Assume for simplicity that the tensor A_0 is isotropic, *i.e.* for all $\boldsymbol{\xi} \in \text{Sym}_2$ one has $A_0 \boldsymbol{\xi} = 2M_0 \boldsymbol{\xi} + L_0(\text{tr } \boldsymbol{\xi})I$, with $M_0, L_0 > 0$. Taking first functions $\Psi \in [C_0^{\infty}(F_0 \cap Q)]^2$ we obtain

$$-M_0 \Delta \mathbf{U} - (L_0 + M_0) \nabla \text{div } \mathbf{U} + \mathbf{u} = P_{\mathfrak{H}} \mathbf{f}, \quad (3.3)$$

$$\mathbf{U}(\mathbf{x}, \cdot) \in [H_{\text{per}}^1(Q)]^2, \quad \mathbf{x} \in \Omega, \quad \mathbf{U}(\mathbf{x}, \mathbf{y}) = \chi(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \Omega, \quad \lambda\text{-a.e. } \mathbf{y} \in F_1, \quad \chi \in L^2(\Omega, \widehat{\mathcal{R}}^0), \quad (3.4)$$

where $P_{\mathfrak{F}}$ is the orthogonal projection operator from $[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2$ onto V . Finally, taking arbitrary $\Psi \in \widetilde{V}$ yields additional equations coupling the framework $F_1 \cap Q$ and the inclusion component $F_0 \cap Q$. For example, on those links that are parallel to the y_2 -axis, we obtain

$$\frac{\theta^2 K_1}{3} \partial_2^4 \chi_1 + (L_0 + 2M_0) \partial_1 U_2 + ((u_0)_1 + \chi_1) = (P_{\mathfrak{F}} \mathbf{f})_1, \quad (3.5)$$

For a general periodic framework F_1 , on each link there is a positively orientated pair of vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ with $\boldsymbol{\tau}$ pointing along the link and $\boldsymbol{\nu}$ orthogonal to the link. The corresponding version of the equation (3.5) on each link of F_1 is as follows:

$$\frac{\theta^2 K_1}{3} \partial_{\boldsymbol{\tau}}^4 \chi^{(\nu)} + (L_0 + 2M_0) \partial_{\boldsymbol{\nu}} U^{(\nu)} + (u_0^{(\nu)} + \chi^{(\nu)}) = (P_{\mathfrak{F}} \mathbf{f})^{(\nu)}, \quad (3.6)$$

where $\partial_{\boldsymbol{\tau}}$, $\partial_{\boldsymbol{\nu}}$ denote differentiation along the link and in the direction normal to the link.

3.2 Extension theorem

Before proving the main result, we recall the description of a class of functions that extend periodic rigid displacements in $\widehat{\mathcal{R}}^0$ on the framework F_1 to the rod network F_1^h , introduced in [20].

Definition 3.2. Let D denote the set of functions $\mathbf{g} \in \widehat{\mathcal{R}}^0$ such that:

1. The function \mathbf{g} is infinitely smooth outside a neighbourhood of the nodes of the network F_1 ;
2. In a neighbourhood $B_{\delta}(\mathcal{O}) := \{\mathbf{y} : |\mathbf{y} - \mathcal{O}| < \delta\}$, $\delta > 0$, of each node \mathcal{O} the function \mathbf{g} takes the form $\mathbf{g}(\mathbf{y}) = C(\boldsymbol{\omega}(\mathbf{y}) - \boldsymbol{\omega}(\mathcal{O}))$, $\mathbf{y} \in F_1$, where C is a constant, $\boldsymbol{\omega}(\mathbf{y}) := (-y_2, y_1)$.

The following two statements are proved in [20].

Proposition 3.1. *The set D is dense in the space $\widehat{\mathcal{R}}^0$ with the respect to the norm of $[L_{\text{per}}^2(Q, d\lambda)]^2$.*

Proposition 3.2. *For each $\mathbf{g} \in D$, there exists a smooth extension $\mathbf{g}^h = \mathbf{g}^h(\mathbf{y})$ to the network F_1^h with the following properties:*

1. For each node \mathcal{O}_k of $F_1 \cap Q$, the symmetric gradient $\mathbf{e}_{\mathbf{y}}(\mathbf{g}^h)$ is zero in $B_{\delta_k}(\mathcal{O}_k)$ for some $\delta_k > 0$.
2. For each $h > 0$ and for each link I of $F_1 \cap Q$ we set

$$\boldsymbol{\sigma}^h(\mathbf{y}) := (h^{-1} \boldsymbol{\nu} \cdot (\mathcal{O} - \mathbf{y}))(\boldsymbol{\tau} \otimes \boldsymbol{\tau}), \quad \mathbf{y} \in I^h \setminus (\cup_k B_{\delta_k}(\mathcal{O}_k)), \quad (3.7)$$

where $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ are the unit tangent and normal to the link I , I^h is the h -neighbourhood of I , \mathcal{O} is either of the two end-points of I , and the union is taken over all nodes of $F_1 \cap Q$. Consider also the “network-to-rod extension” $[(\mathbf{g} \cdot \boldsymbol{\nu})'' K_1]^h$ of the function $(\mathbf{g} \cdot \boldsymbol{\nu})'' K_1$, as in the second formula in (2.8).

Then the asymptotic formula

$$A_1 \mathbf{e}_{\mathbf{y}}(\mathbf{g}^h) = h[(\mathbf{g} \cdot \boldsymbol{\nu})'' K_1]^h \boldsymbol{\sigma}^h + O(h^2), \quad h \rightarrow 0, \quad (3.8)$$

holds on $(F_1^h \cap Q) \setminus (\cup_k B_{\delta_k}(\mathcal{O}_k))$.

3.3 Convergence of solutions

Theorem 3.1. *For all ε, h , let \mathbf{u}_ε^h solve the integral identity (1.1) with right-hand side $\mathbf{f} = \mathbf{f}_\varepsilon^h$, and suppose that $h/\varepsilon \rightarrow \theta > 0$ as $\varepsilon \rightarrow 0$. If $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$ then $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$, and \mathbf{u} satisfies (3.1). If $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$ then $\mathbf{u}_\varepsilon^h \xrightarrow{2} \mathbf{u}$ and, in addition, there is convergence of the corresponding elastic energies.*

Proof. Setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0(\mathbf{x})$ in the identity (1.1) and using Theorems 2.2, 2.4, we obtain

$$\int_{\Omega} A_{\lambda}^{\text{hom}} \mathbf{e}(\mathbf{u}_0) \cdot \mathbf{e}(\boldsymbol{\varphi}_0) \, d\mathbf{x} + \int_{\Omega} \int_Q \mathbf{u} \cdot \boldsymbol{\varphi}_0 \, d\mu d\mathbf{x} = \int_{\Omega} \int_Q \mathbf{f} \cdot \boldsymbol{\varphi}_0 \, d\mu d\mathbf{x}. \quad (3.9)$$

Suppose that $\mathbf{G} \in [C_{\text{per}}^{\infty}(Q)]^2$, $\mathbf{g} \in D$ are such that $\mathbf{G}(\mathbf{y}) = \mathbf{g}(\mathbf{y})$ for all $\mathbf{y} \in F_1 \cap Q$. We approximate the function \mathbf{G} by a sequence $\mathbf{G}^h \in [C_{\text{per}}^{\infty}(Q)]^2$ such that $\mathbf{G}^h = \mathbf{g}^h$ on $F_1^h \cap Q$, where \mathbf{g}^h is the extension described in Proposition 3.2. This is achieved, *e.g.*, by setting $\mathbf{G}^h = \mathbf{G}\chi_h + \bar{\mathbf{g}}^h(1 - \chi_h)$, where

$$\bar{\mathbf{g}}^h(\mathbf{y}) := \begin{cases} \mathbf{0}, & \mathbf{y} \in F_0^{2h} \cap Q, \\ \mathbf{g}^h(\mathbf{y}), & \mathbf{y} \in F_1^{2h} \cap Q, \end{cases}$$

and χ_h is the convolution of the characteristic function of the set $F_0^{3h/2}$ with a function $v(\cdot/h)$ such that $v \in C_0^{\infty}(\mathbb{R}^2)$, $\text{supp}(v) \subset \{\mathbf{z} \in \mathbb{R}^2 : |\mathbf{z}| \leq 1/4\}$.

Lemma 3.1. *For the sequence \mathbf{G}^h constructed above, one has $\|\mathbf{G}^h - \mathbf{G}\|_{[H^1(Q)]^2} \rightarrow 0$ as $h \rightarrow 0$.*

Proof. Note first that since $\mathbf{G}, \mathbf{G}^h, \mathbf{g}^h$ are smooth and therefore their L^2 -norms on F_1^h and $F_1^{2h} \setminus F_1^h$ are of order $O(h)$ as $h \rightarrow 0$, and in view of the fact that $\mathbf{G}^h = \mathbf{G}$ on F_0^{2h} , one has $\|\mathbf{G}^h - \mathbf{G}\|_{[L^2(Q)]^2} \rightarrow 0$ as $h \rightarrow 0$.

Further, since $\mathbf{G}^h - \mathbf{G} = (\bar{\mathbf{g}}^h - \mathbf{G})(1 - \chi_h)$ and by the same argument as above one has

$$\|\mathbf{e}(\bar{\mathbf{g}}^h - \mathbf{G})(1 - \chi_h)\|_{[L^2(Q)]^3} \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

in order to estimate L^2 -norm of $\mathbf{e}(\mathbf{G}^h - \mathbf{G})$ it is sufficient to consider³ $\|(\bar{\mathbf{g}}^h - \mathbf{G}) \otimes \nabla \chi_h\|_{[L^2((F_1^{2h} \setminus F_1^h) \cap Q)]^3}$. To this end, notice that $\nabla \chi_h = O(h^{-1})$, and since $\mathbf{G} = \mathbf{g}^h$ on $F_1 \cap Q$ one has

$$\mathbf{G}(\mathbf{y}) = \mathbf{g}^h(\mathbf{y}) + O(h), \quad h \rightarrow 0, \quad \mathbf{y} \in (F_1^{2h} \setminus F_1^h) \cap Q,$$

uniformly in \mathbf{y} . It follows that

$$\|(\bar{\mathbf{g}}^h - \mathbf{G}) \otimes \nabla \chi_h\|_{[L^2((F_1^{2h} \setminus F_1^h) \cap Q)]^3} \leq Ch, \quad C > 0,$$

from which the claim follows. \square

Taking in (1.1) test functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}^{\varepsilon, h} = w \mathbf{G}^h(\cdot/\varepsilon)$, where $w \in C_0^{\infty}(\Omega)$, yields

$$\begin{aligned} \varepsilon^{-1} \int_{\Omega_1^{\varepsilon, h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_{\mathbf{y}}(\mathbf{g}^h)(\cdot/\varepsilon) w \, d\mu_\varepsilon^h + \int_{\Omega_1^{\varepsilon, h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot (\mathbf{g}^h(\cdot/\varepsilon) \otimes \nabla w) \, d\mu_\varepsilon^h \\ + \varepsilon \int_{\Omega_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_{\mathbf{y}}(\mathbf{G}^h)(\cdot/\varepsilon) w \, d\mu_\varepsilon^h \\ + \varepsilon^2 \int_{\Omega_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot (\mathbf{G}^h(\cdot/\varepsilon) \otimes \nabla w) \, d\mu_\varepsilon^h = \int_{\Omega} (\mathbf{f}_\varepsilon^h - \mathbf{u}_\varepsilon^h) \cdot \mathbf{G}^h(\cdot/\varepsilon) w \, d\mu_\varepsilon^h, \end{aligned} \quad (3.10)$$

³Throughout, we use the notation \otimes for the symmetrised tensor product.

We denote the four terms on the left-hand side of (3.10) by $I_j(\varepsilon)$, $j = 1, 2, 3, 4$. It follows from the L^2 -boundedness of the sequence $\varepsilon \mathbf{e}(\mathbf{u}_\varepsilon^h)$ and the fact that $A_1(\mathbf{e}(\mathbf{u}_0(\mathbf{x})) + \mathbf{v}(\mathbf{x}, \mathbf{y}))$ is pointwise orthogonal to the matrix $\mathbf{g}(\mathbf{y}) \otimes \nabla w(\mathbf{x})$, for $\mathbf{y} \in F_1 \cap Q$, $\mathbf{x} \in \Omega$, (see [15, Lemma 5.3]) that the terms $I_4(\varepsilon)$ and $I_2(\varepsilon)$ converge to zero as $\varepsilon \rightarrow 0$. The convergence results on the soft component discussed in Section 2.5 imply that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_3(\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \varepsilon \int_{\Omega} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_\mathbf{y}(\mathbf{G})(\cdot/\varepsilon) w \, d\mu_\varepsilon^h + \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \varepsilon \int_{\Omega} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_\mathbf{y}(\mathbf{G}^h - \mathbf{G})(\cdot/\varepsilon) w \, d\mu_\varepsilon^h \\ &= \frac{1}{2} \int_{\Omega} \int_Q A_0 \mathbf{e}_\mathbf{y}(\mathbf{U}) \cdot \mathbf{e}_\mathbf{y}(\mathbf{G}) w \, dy dx. \end{aligned}$$

The following statement is a consequence of [20, Lemma 3.5].

Proposition 3.3. *The two-scale convergence*

$$\frac{h}{\varepsilon} \chi_1^{h,\varepsilon} \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \boldsymbol{\sigma}_\varepsilon^h \stackrel{2}{\rightharpoonup} \frac{\theta^2}{3} \chi_1(\mathbf{y})(\boldsymbol{\chi} \cdot \boldsymbol{\nu})'' \quad (3.11)$$

holds, where $\boldsymbol{\sigma}^h$ is the function defined by (3.7).

Taking into account the asymptotics (3.8), it follows from the above proposition that the convergence

$$\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = \frac{\theta^2}{6} \int_{\Omega} \int_Q K_1 \boldsymbol{\chi}'' \cdot \mathbf{g}'' w \, d\lambda dx \quad (3.12)$$

holds. Finally, passing to the limit in (3.10) as $\varepsilon \rightarrow 0$ we obtain

$$\frac{\theta^2}{6} \int_{\Omega} \int_Q K_1 \boldsymbol{\chi}'' \cdot \mathbf{g}'' w \, d\lambda dx + \frac{1}{2} \int_{\Omega} \int_Q A_0 \mathbf{e}_\mathbf{y}(\mathbf{u}) \cdot \mathbf{e}_\mathbf{y}(\mathbf{G}) w \, dy dx = \int_{\Omega} \int_Q (\mathbf{f} - \mathbf{u}) \cdot \mathbf{G} w \, d\mu dx, \quad (3.13)$$

Adding together the identities (3.9) and (3.13) and denoting $\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varphi}_0(\mathbf{x}) + \boldsymbol{\Phi}(\mathbf{x}, \mathbf{y})$, the homogenised formulation (3.1) follows.

In order to prove the strong convergence of solutions when $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$, consider another version of problem (1.1) with right-hand sides $\mathbf{g}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{g}$:

$$\begin{aligned} \mathbf{v}_\varepsilon^h \in [H_0^1(\Omega)]^2, \quad \int_{\Omega_1^{\varepsilon,h}} A_1 \mathbf{e}(\mathbf{v}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{v}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h \\ + \int_{\Omega} \mathbf{v}_\varepsilon^h \cdot \boldsymbol{\varphi} \, d\mu_\varepsilon^h = \int_{\Omega} \mathbf{g}_\varepsilon^h \cdot \boldsymbol{\varphi} \, d\mu_\varepsilon^h \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2. \end{aligned} \quad (3.14)$$

Setting $\boldsymbol{\varphi} = \mathbf{u}_\varepsilon^h$ in the above, $\boldsymbol{\varphi} = \mathbf{v}_\varepsilon^h$ in the original problem (1.1) with $\mathbf{f} = \mathbf{f}_\varepsilon^h$, and then subtracting one from the other yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{u}_\varepsilon^h \cdot \mathbf{g}_\varepsilon^h \, d\mu_\varepsilon^h = \int_{\Omega} \mathbf{v}_\varepsilon^h \cdot \mathbf{f}_\varepsilon^h \, d\mu_\varepsilon^h = \int_{\Omega} \int_Q \mathbf{v} \cdot \mathbf{f} \, d\mu dx = \int_{\Omega} \int_Q \mathbf{u} \cdot \mathbf{g} \, d\mu dx, \quad (3.15)$$

where where \mathbf{v} solves the homogenised equation with the right-hand side \mathbf{g} .

Finally, in order to show the convergence of energies, we set $\boldsymbol{\varphi} = \mathbf{u}_\varepsilon^h$ in (1.1) with $\mathbf{f} = \mathbf{f}_\varepsilon^h$ and use the definition of strong two-scale convergence as well as the identity (3.1), as follows:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\Omega_1^{\varepsilon,h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) \, d\mu_\varepsilon^h + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) \, d\mu_\varepsilon^h \right\} &= \int_{\Omega} \int_Q |\mathbf{f}|^2 \, d\mu dx - \int_{\Omega} \int_Q |\mathbf{u}|^2 \, d\mu dx \\ &= \int_{\Omega} A_\lambda^{\text{hom}} \mathbf{e}(\mathbf{u}_0) \cdot \mathbf{e}(\mathbf{u}_0) \, dx + \frac{\theta^2}{6} \int_{\Omega} \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\chi}'' \, d\lambda dx + \frac{1}{2} \int_{\Omega} \int_Q A_0 \mathbf{e}_\mathbf{y}(\mathbf{U}) \cdot \mathbf{e}_\mathbf{y}(\mathbf{U}) \, dy dx. \end{aligned}$$

□

4 Convergence of spectra

Here we establish the convergence of the spectra of the operators associated with (1.1) to the spectrum given by the limit problem (3.1).

4.1 Spectrum of the limit operator

Consider the bilinear forms (*cf.* (3.1))

$$\mathbf{b}_{\text{macro}}(\mathbf{u}_0, \varphi_0) = \int_{\Omega} A_{\lambda}^{\text{hom}} \mathbf{e}(\mathbf{u}_0) \cdot \mathbf{e}(\varphi_0) \, d\mathbf{x}, \quad \mathbf{u}_0, \varphi_0 \in [H_0^1(\Omega)]^2, \quad (4.1)$$

$$\mathbf{b}_{\text{micro}}(\mathbf{U}, \Phi) = \frac{\theta^2}{6} \int_Q K_1 \chi'' \cdot \Phi'' \, d\lambda + \frac{1}{2} \int_Q A_0 \mathbf{e}_{\mathbf{y}}(\mathbf{U}) \cdot \mathbf{e}_{\mathbf{y}}(\Phi) \, d\mathbf{y}, \quad \mathbf{U}, \Phi \in \tilde{V}, \quad (4.2)$$

where the space \tilde{V} is defined in Section 3.1, see the paragraph preceding (3.3). The spectral problem associated with (3.1) can be written in the form

$$\begin{aligned} \mathbf{b}_{\text{macro}}(\mathbf{u}_0, \varphi_0) &= s(\mathbf{u}_0 + \langle \mathbf{U} \rangle, \varphi_0)_{[L^2(\Omega)]^2} \quad \forall \varphi_0 \in [H_0^1(\Omega)]^2, \\ \mathbf{b}_{\text{micro}}(\mathbf{U}, \Phi) &= s(\mathbf{u}_0 + \mathbf{U}, \Phi)_{[L^2(Q, d\mu)]^2} \quad \forall \Phi \in \tilde{V}. \end{aligned} \quad (4.3)$$

Let $\{\phi_n\}_{n \in \mathbb{N}} \subset \tilde{V}$ be an orthonormal set of eigenvectors with non-zero average for the bilinear form $\mathbf{b}_{\text{micro}}$ with corresponding set of eigenvalues $\{\omega_n\}_{n \in \mathbb{N}}$:

$$\mathbf{b}_{\text{micro}}(\phi_n, \Phi) = \omega_n (\phi_n, \Phi)_{[L^2(Q, d\mu)]^2} \quad \forall \Phi \in \tilde{V}. \quad (4.4)$$

Assuming that the value s is outside the spectrum $\text{Sp}(\mathbf{b}_{\text{micro}})$ of the form $\mathbf{b}_{\text{micro}}$, the function $\mathbf{U}(\mathbf{x}, \mathbf{y})$ is written as a series in terms of eigenfunctions $\{\phi_n\}_{n \in \mathbb{N}}$:

$$\mathbf{U}(\mathbf{x}, \mathbf{y}) = s \sum_{n=1}^{\infty} \frac{\langle \phi_n \rangle \cdot \mathbf{u}_0(\mathbf{x})}{\omega_n - s} \phi_n(\mathbf{y}). \quad (4.5)$$

Substituting this expansion for \mathbf{U} into (4.3), we obtain

$$\mathbf{b}_{\text{macro}}(\mathbf{u}_0, \varphi_0) = (\beta(s) \mathbf{u}_0, \varphi_0)_{[L^2(\Omega)]^2} \quad \forall \varphi_0 \in [H_0^1(\Omega)]^2, \quad \beta(s) := s \left(I + s \sum_{n=1}^{\infty} \frac{\langle \phi_n \rangle \otimes \langle \phi_n \rangle}{\omega_n - s} \right). \quad (4.6)$$

Versions of the function β appear in the study of scalar [15] and vector ([14], [21], [22]) homogenisation problems. The following statement is a straightforward modification of a result in [22].

Proposition 4.1. *Consider the operator \mathfrak{A} whose domain consists of all solution pairs $(\mathbf{u}_0, \mathbf{U})$ for the identity*

$$\mathbf{b}_{\text{macro}}(\mathbf{u}_0, \varphi_0) + \mathbf{b}_{\text{micro}}(\mathbf{U}, \Phi) = (\mathbf{f}, \varphi_0 + \Phi)_{[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2} \quad \forall \varphi_0 + \Phi \in V, \quad (4.7)$$

as the right-hand side \mathbf{f} runs over all elements of \mathfrak{H} and defined by $\mathbf{f} = \mathfrak{A}(\mathbf{u}_0 + \mathbf{U})$ if and only if (4.7) holds. Then the resolvent set $\rho(\mathfrak{A})$ of the operator \mathfrak{A} is given by

$$\rho(\mathfrak{A}) = \rho(\mathbf{b}_{\text{micro}}) \cap \{s \mid \text{all eigenvalues of } \beta(s) \text{ belong to } \rho(\mathbf{b}_{\text{macro}})\}, \quad (4.8)$$

where $\rho(\mathbf{b}_{\text{micro}})$ is the resolvent set of the operator generated by the form $\mathbf{b}_{\text{micro}}$ in the closure⁴ of \tilde{V} in $[L^2(Q)]^2$, and $\rho(\mathbf{b}_{\text{macro}})$ is the resolvent set of the operator generated by the form $\mathbf{b}_{\text{macro}}$.

⁴Note that the domain of this operator is dense in this closure.

Proof. Suppose that s belongs to the right-hand side of (4.8). We argue that the problem

$$\begin{cases} \mathbf{b}_{\text{macro}}(\mathbf{u}_0, \varphi_0) - s(\mathbf{u}_0 + \langle \mathbf{U} \rangle, \varphi_0)_{[L^2(\Omega)]^2} = (\mathbf{f}, \varphi_0)_{[L^2(\Omega)]^2} & \forall \varphi_0, \\ \mathbf{b}_{\text{micro}}(\mathbf{U}, \Phi) - s(\mathbf{u}_0 + \mathbf{U}, \Phi)_{[L^2(Q, d\mu)]^2} = (\mathbf{f}, \Phi)_{[L^2(Q, d\mu)]^2} & \forall \Phi. \end{cases} \quad (4.9)$$

has a solution for every $\mathbf{f} \in \mathfrak{H}$, given that s satisfies the required assumptions of the lemma. Since $s \notin \text{Sp}(\mathbf{b}_{\text{micro}})$, it follows that \mathbf{U} can be written in the form (4.5) with \mathbf{u}_0 replaced by $s\mathbf{u}_0 + \mathbf{f}$. Substituting this into the first equation of (4.9) yields

$$\mathbf{b}_{\text{macro}}(\mathbf{u}_0, \varphi_0) - (\beta(s)\mathbf{u}_0, \varphi_0)_{[L^2(\Omega)]^2} = (s^{-1}\beta(s)\mathbf{f}, \varphi_0)_{[L^2(\Omega)]^2} \quad \forall \varphi_0. \quad (4.10)$$

Since all eigenvalues of $\beta(s)$ are in $\rho(\mathbf{b}_{\text{macro}})$, the operator induced by the bilinear form on the left-hand side of (4.10) is invertible and thus the identity (4.10) has a unique solution.

Conversely, one has $\rho(\mathfrak{A}) \subset \rho(\mathbf{b}_{\text{micro}})$ and if $s \in \rho(\mathfrak{A})$ then $\beta(s)$ has no eigenvalues in $\text{Sp}(\mathbf{b}_{\text{macro}})$, for otherwise the problem (4.9) would not be uniquely solvable for any $\mathbf{f} \in \mathfrak{H}$. \square

In the case of the model framework, the matrix β is proportional to the identity matrix I . Indeed, if the set $F_1 \cap Q$ is invariant with respect to a rotation \mathbf{R} , *i.e.* one has $\{\mathbf{R}\mathbf{y} : \mathbf{y} \in F_1 \cap Q\} = F_1 \cap Q$, then for an eigenfunction ϕ of the bilinear form $\mathbf{b}_{\text{micro}}$, the vector $\mathbf{R}\phi$ is an eigenvector with the same eigenvalue, hence one has $\mathbf{R}\beta(s)\mathbf{R}^{-1} = \beta(s)$, in view of the definition of β , see (4.6). Taking \mathbf{R} to be the rotation through $\pi/2$ yields the required claim, namely $\beta(s) = b(s)I$ for a scalar function b . Let $\{\gamma_n\}_{n \in \mathbb{N}}$ denote the increasing sequence of zeros of the function b and let $\{\delta_n\}_{n \in \mathbb{N}}$ be the increasing sequence of all eigenvalues in the set $\{\omega_n\}_{n \in \mathbb{N}}$, counting multiple eigenvalues only once. The spectrum of the limit operator \mathfrak{A} has the “band” form:

$$\text{Sp}(\mathfrak{A}) = \left(\bigcup_{n \in \mathbb{N}} \{s \in (\gamma_n, \delta_n) : b(s) \in \text{Sp}(\mathbf{b}_{\text{macro}})\} \right) \cup \{\delta_n\}_{n \in \mathbb{N}} \cup \{\alpha_n\}_{n \in \mathbb{N}},$$

where α_n are the eigenvalues of $\mathbf{b}_{\text{micro}}$ such that all of the corresponding eigenfunctions have zero average over Q . The intervals (δ_n, γ_{n+1}) , $n \in \mathbb{N}$, are “gaps” in the spectrum, which do not have common points with $\text{Sp}(\mathfrak{A})$, except, possibly, for elements of the set $\{\alpha_n\}_{n \in \mathbb{N}}$.

4.2 Proof of spectral convergence

Here we show that the spectra of the original problems converge to the spectrum of the limit problem (3.1).

Definition 4.1. We say that a sequence of sets $\mathcal{X}_\varepsilon \subset \mathbb{R}$, $\varepsilon > 0$, converges in the sense of Hausdorff to $\mathcal{X} \subset \mathbb{R}$ if the following two statements hold:

- (H1) For each $\omega \in \mathcal{X}$, there exists a sequence $\omega_\varepsilon \in \mathcal{X}_\varepsilon$ such that $\omega_\varepsilon \rightarrow \omega$;
- (H2) For all sequences $\omega_\varepsilon \in \mathcal{X}_\varepsilon$ such that $\omega_\varepsilon \rightarrow \omega \in \mathbb{R}$, it follows that $\omega \in \mathcal{X}$.

Definition 4.2. We say that a family of operators \mathcal{A}_ε in $[L^2(\Omega, d\mu_\varepsilon^h)]^2$ strongly two-scale resolvent convergent as $\varepsilon \rightarrow 0$ to an operator \mathcal{A} in $[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2$, and write $\mathcal{A}_\varepsilon \xrightarrow{2} \mathcal{A}$, if for all \mathbf{f} in the range $R(\mathcal{A})$ of the operator \mathcal{A} and for all sequences $\mathbf{f}_\varepsilon^h \in [L^2(\Omega, d\mu_\varepsilon^h)]^2$ such that $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$, the two-scale convergence $(\mathcal{A}_\varepsilon + I)^{-1} \mathbf{f}_\varepsilon^h \xrightarrow{2} (\mathcal{A} + I)^{-1} \mathbf{f}$ holds.

Proposition 4.2. If $\mathcal{A}_\varepsilon \xrightarrow{2} \mathcal{A}$, then the property (H1) holds with $\mathcal{X}_\varepsilon = \text{Sp}(\mathcal{A}_\varepsilon)$, $\mathcal{X} = \text{Sp}(\mathcal{A})$.

Proof. Let $T_\varepsilon := (\mathcal{A}_\varepsilon + I)^{-1}$ and $T := (\mathcal{A} + I)^{-1}$. If $s \in \text{Sp}(\mathcal{A})$ then $t = (1 + s)^{-1} \in \text{Sp}(T)$. Therefore, for any $\delta > 0$, there exists a vector $\mathbf{f} \in R(\mathcal{A})$ such that

$$\|\mathbf{f}\|_{[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2} = 1, \quad \|(T - t)\mathbf{f}\|_{[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2} \leq \delta/4.$$

Consider a sequence $\mathbf{f}_\varepsilon^h \in [L^2(\Omega, d\mu_\varepsilon^h)]^2$ such that $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$. Using the definition of strong two-scale resolvent convergence, one has

$$\lim_{\varepsilon \rightarrow 0} \|(T_\varepsilon - t)\mathbf{f}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} = \|(T - t)\mathbf{f}\|_{[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2} \leq \delta/4.$$

Hence, $\|(T_\varepsilon - t)\mathbf{f}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} \leq \delta/2$ and $\|\mathbf{f}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} \geq 1/2$ for sufficiently small ε . Therefore, the interval $(t - \delta, t + \delta)$ contains a point of the spectrum of the operator T_ε . Moreover, every interval centered at s contains a point of the spectrum of the operator \mathcal{A}_ε for small enough ε , which completes the proof. \square

Corollary 4.1. *For the operators $\mathfrak{A}_\varepsilon^h$ defined by the identity*

$$\mathfrak{B}_\varepsilon^h(\mathbf{u}, \mathbf{v}) = \mathfrak{L}_\varepsilon^h(\mathbf{v}),$$

where the forms $\mathfrak{B}_\varepsilon^h, \mathfrak{L}_\varepsilon^h$ are defined by (1.2), $\mathbf{f} = \mathfrak{A}_\varepsilon^h \mathbf{u}$, and the operator \mathfrak{A} is defined in proposition 4.1, the property (H1) holds with $\mathcal{X}_\varepsilon = \text{Sp}(\mathfrak{A}_\varepsilon^h)$, $\mathcal{X} = \text{Sp}(\mathfrak{A})$, $h = h(\varepsilon)$.

The property (H2) of the Hausdorff convergence does not hold for spectra $\text{Sp}(\mathfrak{A}_\varepsilon^h)$ in general, due to the fact that the soft component may have a non-empty intersection with the boundary of Ω . However, a suitable version of (H2) does hold for a modified operator family, where the corresponding elements of the soft component are replaced by the stiff material. More precisely, for each ε, h , denote by $\widehat{\mathfrak{A}}_\varepsilon^h$ the operator defined similarly to $\mathfrak{A}_\varepsilon^h$, with $\Omega_0^{\varepsilon, h}$ and $\Omega_1^{\varepsilon, h}$ in (1.1) replaced by $\widehat{\Omega}_0^{\varepsilon, h}$ and $\Omega \setminus \widehat{\Omega}_0^{\varepsilon, h}$. Here, the set $\widehat{\Omega}_0^{\varepsilon, h}$ is the union of the sets $\varepsilon(F_0 \cap Q^h + \mathbf{n})$ over all $\mathbf{n} \in \mathbb{Z}^2$ such that $\varepsilon(Q + \mathbf{n}) \subset \Omega$.

Theorem 4.1. *Suppose that of all ε, h , the function $\mathbf{u}_\varepsilon^h \in [H_0^1(\Omega)]^2$ is the L^2 -normalised eigenfunction of $\widehat{\mathfrak{A}}_\varepsilon$:*

$$\widehat{\mathfrak{A}}_\varepsilon \mathbf{u}_\varepsilon^h = \omega_\varepsilon \mathbf{u}_\varepsilon^h, \quad \|\mathbf{u}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} = 1. \quad (4.11)$$

If $\omega_\varepsilon \rightarrow \omega \notin \text{Sp}(\mathbf{b}_{\text{micro}})$, then the eigenfunction sequence $\{\mathbf{u}_\varepsilon^h\}$ is compact with respect to strong two-scale convergence in $[L^2(\Omega, d\mu_\varepsilon^h)]^2$.

Proof. The eigenvalue problem (4.11) is understood in the sense of the identity

$$\int_{\Omega \setminus \widehat{\Omega}_0^{\varepsilon, h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) d\mu_\varepsilon^h + \varepsilon^2 \int_{\widehat{\Omega}_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) d\mu_\varepsilon^h = \omega_\varepsilon \int_{\Omega} \mathbf{u}_\varepsilon^h \cdot \boldsymbol{\varphi} d\mu_\varepsilon^h \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2, \quad (4.12)$$

which implies, in particular, that

$$\int_{\Omega \setminus \widehat{\Omega}_0^{\varepsilon, h}} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) d\mu_\varepsilon^h + \varepsilon^2 \int_{\widehat{\Omega}_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h) d\mu_\varepsilon^h = \omega_\varepsilon.$$

and hence $\|\mathbf{e}(\mathbf{u}_\varepsilon^h)\|_{[L^2(\widehat{\Omega}_1^{\varepsilon, h}, d\mu_\varepsilon^h)]^2}$ are uniformly bounded. Denote by $\widehat{\Omega}_1^{\varepsilon, h}$ the union of $\varepsilon(F_1 \cap Q^h + \mathbf{n})$ over all $\mathbf{n} \in \mathbb{Z}^2$ such that $\varepsilon(Q + \mathbf{n}) \subset \Omega$. We claim that for all ε, h , there exists $\widetilde{\mathbf{u}}_\varepsilon^h$ such that

$$\mathbf{e}(\mathbf{u}_\varepsilon^h) = \mathbf{e}(\widetilde{\mathbf{u}}_\varepsilon^h) \text{ on } \widehat{\Omega}_1^{\varepsilon, h}, \quad \widetilde{\mathbf{u}}_\varepsilon^h \in [H_0^1(\Omega)]^2, \quad \|\mathbf{e}(\widetilde{\mathbf{u}}_\varepsilon^h)\|_{[L^2(\widehat{\Omega}_0^{\varepsilon, h}, d\mu_\varepsilon^h)]^2} \leq C \|\mathbf{e}(\mathbf{u}_\varepsilon^h)\|_{[L^2(\widehat{\Omega}_1^{\varepsilon, h}, d\mu_\varepsilon^h)]^2}, \quad (4.13)$$

$$\int_{\widehat{\Omega}_0^{\varepsilon, h}} A_0 \mathbf{e}(\widetilde{\mathbf{u}}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) d\mu_\varepsilon^h = 0 \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2 \text{ such that } \mathbf{e}(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{\varepsilon, h}, \quad (4.14)$$

where the constant $C > 0$ is independent of ε, h . Indeed, we can consider $\widetilde{\mathbf{u}}_\varepsilon^h$ such that $\mathbf{z}_\varepsilon^h := \mathbf{u}_\varepsilon^h - \widetilde{\mathbf{u}}_\varepsilon^h$ solves the minimisation problem

$$\frac{1}{2} \int_{\widehat{\Omega}_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{v}) \cdot \mathbf{e}(\mathbf{v}) d\mu_\varepsilon^h - \int_{\widehat{\Omega}_0^{\varepsilon, h}} A_0 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\mathbf{v}) d\mu_\varepsilon^h \mapsto \min, \quad (4.15)$$

over all functions $\mathbf{v} \in [H_0^1(\Omega)]^2$ whose restriction to $\widehat{\Omega}_1^{h,\varepsilon}$ is a rigid-body motion with respect to the Lebesgue measure, *i.e.* one has $\mathbf{e}(\mathbf{v}) = 0$ in $\widehat{\Omega}_1^{h,\varepsilon}$. Clearly, one has $\mathbf{e}(\mathbf{z}_\varepsilon^h) = 0$ in $\widehat{\Omega}_1^{\varepsilon,h}$ and

$$\begin{aligned} & \int_{\Omega \setminus (\widehat{\Omega}_0^{\varepsilon,h} \cup \widehat{\Omega}_1^{\varepsilon,h})} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h + \int_{\widehat{\Omega}_1^{\varepsilon,h}} A_1 \mathbf{e}(\mathbf{z}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h + \varepsilon^2 \int_{\widehat{\Omega}_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{z}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h \\ & - \omega_\varepsilon \int_{\Omega} \mathbf{z}_\varepsilon^h \cdot \boldsymbol{\varphi} \, d\mu_\varepsilon^h = \omega_\varepsilon \int_{\Omega} \widetilde{\mathbf{u}}_\varepsilon^h \cdot \boldsymbol{\varphi} \, d\mu_\varepsilon^h \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2, \quad \mathbf{e}(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{\varepsilon,h}, \end{aligned} \quad (4.16)$$

by combining (4.12), (4.14) and the Euler-Lagrange equation for (4.15). It follows from the bound (4.13) that $\widetilde{\mathbf{u}}_\varepsilon^h$ is compact with respect to strong convergence in $[L^2(\Omega, d\mu_\varepsilon^h)]^2$: there exists $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}(\mathbf{x})$ such that, up to selecting a subsequence, one has $\widetilde{\mathbf{u}}_\varepsilon^h \rightarrow \widetilde{\mathbf{u}}$ in $[L^2(\Omega, d\mu_\varepsilon^h)]^2$.

Lemma 4.1. *Suppose that for each ε, h the function \mathbf{f}_ε^h belongs to the closure in $[L^2(\Omega)]^2$ of the set of smooth functions whose restrictions to $\widehat{\Omega}_1^{\varepsilon,h}$ are rigid-body motions with respect to the Lebesgue measure. Suppose also that $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$ in \mathfrak{H} , where the space \mathfrak{H} is given in Definition 3.1.*

For all ε, h , consider the function $\mathbf{v}_\varepsilon^h \in [H_0^1(\Omega)]^2$ such that $\mathbf{e}(\mathbf{v}_\varepsilon^h) = 0$ in $\widehat{\Omega}_1^{\varepsilon,h}$ and the following resolvent identity holds (cf. (4.16)):

$$\begin{aligned} & \int_{\Omega \setminus (\widehat{\Omega}_0^{\varepsilon,h} \cup \widehat{\Omega}_1^{\varepsilon,h})} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h + \int_{\widehat{\Omega}_1^{\varepsilon,h}} A_1 \mathbf{e}(\mathbf{v}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h + \varepsilon^2 \int_{\widehat{\Omega}_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{v}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h \\ & - \omega_\varepsilon \int_{\Omega} \mathbf{v}_\varepsilon^h \cdot \boldsymbol{\varphi} \, d\mu_\varepsilon^h = \int_{\Omega} \mathbf{f}_\varepsilon^h \cdot \boldsymbol{\varphi} \, d\mu_\varepsilon^h \quad \forall \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2, \quad \mathbf{e}(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{\varepsilon,h}. \end{aligned} \quad (4.17)$$

Then $\mathbf{v}_\varepsilon^h \xrightarrow{2} \mathbf{v} = \mathbf{v}(\mathbf{x}, \mathbf{y}) \in [L^2(\Omega, \widetilde{V})]^2$, and

$$\begin{aligned} & \frac{\theta^2}{6} \int_{\Omega} \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\Phi}'' \, d\lambda(\mathbf{y}) \, d\mathbf{x} + \frac{1}{2} \int_{\Omega} \int_Q A_0 \mathbf{e}_{\mathbf{y}}(\mathbf{v}) \cdot \mathbf{e}_{\mathbf{y}}(\boldsymbol{\varphi}) \, d\mathbf{y} \, d\mathbf{x} - \omega \int_{\Omega} \int_Q \mathbf{v} \cdot \boldsymbol{\varphi} \, d\mu(\mathbf{y}) \, d\mathbf{x} \\ & = \int_{\Omega} \int_Q \mathbf{f} \cdot \boldsymbol{\varphi} \, d\mu(\mathbf{y}) \, d\mathbf{x} \quad \forall \boldsymbol{\varphi} \in [L^2(\Omega, \widetilde{V})]^2, \quad \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}) \text{ a.e } \mathbf{x} \in \Omega, \text{ } \lambda\text{-a.e } \mathbf{y} \in F_1 \cap Q, \end{aligned} \quad (4.18)$$

where $\boldsymbol{\chi}(\mathbf{x}, \cdot)$ is the trace of the function $\mathbf{v}(\mathbf{x}, \cdot)$ on $F_1 \cap Q$ for a.e. $\mathbf{x} \in \Omega$.

Proof. We show first that the spectra of the operators $\widehat{\mathfrak{A}}_\varepsilon^0$ defined via the bilinear forms (cf. (4.17))

$$\begin{aligned} \widehat{\mathfrak{b}}_\varepsilon^0(\mathbf{v}, \boldsymbol{\varphi}) &= \int_{\Omega \setminus (\widehat{\Omega}_0^{\varepsilon,h} \cup \widehat{\Omega}_1^{\varepsilon,h})} A_1 \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h + \int_{\widehat{\Omega}_1^{\varepsilon,h}} A_1 \mathbf{e}(\mathbf{z}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h \\ &+ \varepsilon^2 \int_{\widehat{\Omega}_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{v}) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu_\varepsilon^h \quad \mathbf{v}, \boldsymbol{\varphi} \in [H_0^1(\Omega)]^2, \quad \mathbf{e}(\mathbf{v}), \mathbf{e}(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{\varepsilon,h}, \end{aligned}$$

converge, in the sense of Hausdorff as $\varepsilon \rightarrow 0$, to $\text{Sp}(\mathbf{b}_{\text{micro}})$. Indeed, the convergence $\widehat{\mathfrak{A}}_\varepsilon^0 \xrightarrow{2} \widehat{\mathfrak{A}}^0$ holds, where the operator $\widehat{\mathfrak{A}}^0$ is associated with the bilinear form

$$\begin{aligned} \widehat{\mathfrak{b}}^0(\mathbf{v}, \boldsymbol{\varphi}) &= \frac{\theta^2}{6} \int_Q K_1 \boldsymbol{\chi}'' \cdot \boldsymbol{\Phi}'' \, d\lambda + \frac{1}{2} \int_{F_0 \cap Q} A_0 \mathbf{e}(\mathbf{v}) \cdot \mathbf{e}(\boldsymbol{\varphi}) \, d\mu \quad \mathbf{v}, \boldsymbol{\varphi} \in [L^2(\Omega, \widetilde{V})]^2, \\ \mathbf{v}(\mathbf{y}) &= \boldsymbol{\chi}(\mathbf{y}), \quad \boldsymbol{\varphi}(\mathbf{y}) = \boldsymbol{\Phi}(\mathbf{y}), \quad \lambda\text{-a.e } \mathbf{y} \in F_1 \cap Q, \end{aligned}$$

and hence, $\text{Sp}(\widehat{\mathfrak{A}}^0) \subset \lim_\varepsilon \text{Sp}(\widehat{\mathfrak{A}}_\varepsilon^0)$ by Proposition 4.2. On the other hand any sequence of L^2 -normalised eigenfunctions of $\widehat{\mathfrak{A}}_\varepsilon^0$ whose eigenvalues ω_ε^0 converge to $\omega^0 \in \mathbb{R}$ is compact in the sense of two-scale

convergence, thanks to [16, Theorem 12.2], and therefore $\omega \in \text{Sp}(\widehat{\mathfrak{A}}^0)$. Finally, notice that $\text{Sp}(\widehat{\mathfrak{A}}^0) = \text{Sp}(\mathfrak{b}_{\text{micro}})$.

It follows that whenever ω_ε in (4.17) converge to a point outside $\text{Sp}(\mathfrak{b}_{\text{micro}})$, the identity (4.17) does not have non-zero solutions \mathbf{v}_ε^h for $\mathbf{f}_\varepsilon^h = 0$ and ω_ε replaced by any value in some finite neighbourhood of the set $\{\omega_\varepsilon\}_{\varepsilon < \varepsilon_0}$ for some $\varepsilon_0 > 0$. Hence, for an L^2 -bounded sequence of the right-hand sides \mathbf{f}_ε^h , the functions \mathbf{v}_ε^h that satisfy (4.17) are uniformly bounded in $[L^2(\Omega, d\mu_\varepsilon^h)]^2$ for $\varepsilon < \varepsilon_0$.

Further, setting $\varphi = \mathbf{v}_\varepsilon^h$ in (4.17) and using the fact that A_0 is positive definite yields the uniform estimate

$$\varepsilon \|\chi_0^{\varepsilon, h} \mathbf{e}(\mathbf{v}_\varepsilon^h)\|_{[L^2(\Omega_0^{\varepsilon, h}, d\mu_\varepsilon^h)]^3} \leq C,$$

for some positive constant C . Proceeding as in Section 2, and using the fact that $\widehat{\Omega}_0^{\varepsilon, h} \cup \widehat{\Omega}_1^{\varepsilon, h} \rightarrow \Omega$ as $\varepsilon \rightarrow 0$, we extract a subsequence of \mathbf{v}_ε^h that weakly two-scale converges to a function $\mathbf{v} \in [L^2(\Omega, \widetilde{V})]^2$ and such that $\chi_0^{\varepsilon, h} \mathbf{e}(\mathbf{v}_\varepsilon^h) \xrightarrow{2} \mathbf{e}_\mathbf{y}(\mathbf{v})$ in $[L^2(\Omega, d\mu_\varepsilon^h)]^3$.

Finally, passing to the limit as $\varepsilon \rightarrow 0$ in (4.17) yields the identity (4.18). By the uniqueness of solution to (4.17), the whole sequence \mathbf{v}_ε^h weakly two-scale converges to \mathbf{v} . \square

Lemma 4.1 implies that the sequence \mathbf{z}_ε^h is compact with respect to weak two-scale convergence, its two-scale limit $\mathbf{z} = \mathbf{z}(\mathbf{x}, \mathbf{y})$ is a rigid-body motion on F_1 and satisfies the weak problem

$$\frac{\theta^2}{6} \int_\Omega \int_Q K_1 \mathbf{v}'' \cdot \Phi'' d\lambda(\mathbf{y}) d\mathbf{x} + \frac{1}{2} \int_\Omega \int_Q A_0 \mathbf{e}_\mathbf{y}(\mathbf{z}) \cdot \mathbf{e}_\mathbf{y}(\varphi) d\mathbf{y} d\mathbf{x} - \omega \int_\Omega \int_Q \mathbf{z} \cdot \varphi d\mathbf{y} d\mathbf{x} = \omega \int_\Omega \int_Q \widetilde{\mathbf{u}} \cdot \varphi d\mu(\mathbf{y}) d\mathbf{x}$$

$$\forall \varphi \in [L^2(\Omega, \widetilde{V})]^2, \quad \mathbf{z}(\mathbf{x}, \mathbf{y}) = \mathbf{v}(\mathbf{x}, \mathbf{y}), \quad \varphi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}), \quad \text{a.e } \mathbf{x} \in \Omega, \quad \lambda\text{-a.e } \mathbf{y} \in F_1 \cap Q, \quad (4.19)$$

Setting $\varphi = \mathbf{v}_\varepsilon^h$ in the identity (4.16) and $\varphi = \mathbf{z}_\varepsilon^h$ in (4.17) yields

$$\int_\Omega \mathbf{z}_\varepsilon^h \cdot \mathbf{f}_\varepsilon^h d\mu_\varepsilon^h = \omega_\varepsilon \int_\Omega \mathbf{v}_\varepsilon^h \cdot \widetilde{\mathbf{u}}_\varepsilon^h d\mu_\varepsilon^h \quad \forall \varepsilon, h. \quad (4.20)$$

Taking the limit of both sides (4.20) as $\varepsilon \rightarrow 0$, $h = h(\varepsilon)$, and using the convergence properties of \mathbf{v}_ε^h , \mathbf{u}_ε^h , we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega \mathbf{z}_\varepsilon^h \cdot \mathbf{f}_\varepsilon^h d\mu_\varepsilon^h = \omega \int_\Omega \int_Q \mathbf{v}(\mathbf{x}, \mathbf{y}) \cdot \widetilde{\mathbf{u}}(\mathbf{x}) d\mu(\mathbf{y}) d\mathbf{x}. \quad (4.21)$$

Further, using (4.18) with $\varphi = \mathbf{z}$, and (4.19) with $\varphi = \mathbf{v}$, we obtain

$$\omega \int_\Omega \int_Q \mathbf{v}(\mathbf{x}, \mathbf{y}) \cdot \widetilde{\mathbf{u}}(\mathbf{x}) d\mu(\mathbf{y}) d\mathbf{x} = \int_\Omega \int_Q \mathbf{f}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{z}(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) d\mathbf{x} \quad (4.22)$$

Finally, setting $\mathbf{f}_\varepsilon^h = \mathbf{z}_\varepsilon^h$ in (4.21) and using (4.22), we infer that $\|\mathbf{z}_\varepsilon^h\|_{[L^2(\Omega, d\mu_\varepsilon^h)]^2} \rightarrow \|\mathbf{z}\|_{[L^2(\Omega \times Q, d\mathbf{x} \times d\mu)]^2}$. Therefore, the sequence \mathbf{z}_ε^h strongly two-scale converges to \mathbf{z} , see Proposition 2.3. \square

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